

# UNRAMIFIEDNESS OF GALOIS REPRESENTATIONS ARISING FROM HILBERT MODULAR SURFACES

MATTHEW EMERTON, DAVIDE A. REDUZZI, AND LIANG XIAO

ABSTRACT. Let  $p$  be a prime number and  $F$  a totally real number field. For each prime  $\mathfrak{p}$  of  $F$  above  $p$  we construct a Hecke operator  $T_{\mathfrak{p}}$  acting on  $(\bmod p^m)$  Katz Hilbert modular classes which agrees with the classical Hecke operator at  $\mathfrak{p}$  for global sections that lift to characteristic zero. Using these operators and the techniques of patching complexes of F. Calegari and D. Geraghty we prove that the Galois representations arising from torsion Hilbert modular classes of parallel weight  $\mathbf{1}$  are unramified at  $p$  when  $[F : \mathbb{Q}] = 2$ . Some partial and some conjectural results are obtained when  $[F : \mathbb{Q}] > 2$ .

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## 1. INTRODUCTION

With this paper we begin the investigation of local-global compatibility results for Galois representations attached to *torsion* classes (of non-cohomological weight) occurring in the *coherent* cohomology of Hilbert modular schemes. The existence of such representations was previously proved in [ERX13<sup>+</sup>] (see also generalizations in [RX14<sup>+</sup>, Bo15<sup>+</sup>, GKo15<sup>+</sup>]). Our results are motivated by the conjectures made in [CG12<sup>+</sup>] by F. Calegari and D. Geraghty, and by the consequent applications to modularity lifting theorems as in *op.cit.*.

Let  $F$  be a totally real number field of degree  $g$  and let  $p$  be a prime number. Denote by  $\mathcal{O}$  the ring of integers in a large enough finite extension of  $\mathbb{Q}_p$  and let  $\varpi$  be a uniformizer in  $\mathcal{O}$ . Fix a suitable integral ideal  $\mathcal{N} \subset \mathcal{O}_F$  coprime with  $p$  and let  $\mathcal{M}$  denote the corresponding Hilbert modular scheme (cf. 2.2), *i.e.*, the fine moduli scheme over  $\mathrm{Spec} \mathcal{O}$  classifying  $g$ -dimensional abelian

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schemes endowed with a principal polarization, an endomorphism action by  $\mathcal{O}_F$ , and a suitable  $\mathcal{N}$ -level structure<sup>1</sup>. We let  $\pi : \mathcal{A} \rightarrow \mathcal{M}$  denote the universal abelian scheme over  $\mathcal{M}$  and we set  $\omega := \pi_* \Omega_{\mathcal{A}/\mathcal{M}}^1$ .

The *coherent* sheaf cohomology of  $\mathcal{M}_{\mathcal{O}/\varpi^m}$  with coefficients in the automorphic line bundles  $\omega^\kappa$  attached to paritious weights  $\kappa = (k_1, \dots, k_g) : \text{Res}_{F/\mathbb{Q}} \mathbb{G}_{m/F} \rightarrow \mathbb{G}_{m/\mathbb{Q}}$  (cf. 2.4) is naturally endowed with an action of commuting Hecke operators  $T_{\mathfrak{q}}$  parametrized by the prime ideals  $\mathfrak{q}$  of  $F$  coprime with  $p\mathcal{N}$ . As proved in [ERX13<sup>+</sup>], Hecke eigenclasses arising from any degree of (coherent) cohomology have canonically attached pseudo-representations of  $\text{Gal}(\overline{F}/F)$  unramified outside  $p\mathcal{N}$  whose Frobenii traces match the Hecke eigenvalues. The existence of such representations follows from classical results when the automorphic weight  $\kappa$  is large enough, since the corresponding modular forms lift to characteristic zero by the ampleness of the sheaf  $\omega^{\mathbf{1}}$  on the minimal compactification of  $\mathcal{M}$  (here  $\mathbf{1} := (1, \dots, 1)$ ). On the other hand, when the weight  $\kappa$  is small, such classes might not lift to characteristic zero, and Galois pseudo-representations are constructed in *op.cit* by exploiting the stratification of  $\mathcal{M}_{\overline{\mathbb{F}}_p}$  induced by partial Hasse invariants, and by constructing trivializations of suitable automorphic line bundles restricted to such strata. (We recall that the case of torsion in Betti cohomology is extensively treated in the groundbreaking work of P. Scholze [Sc15]).

In [CG12<sup>+</sup>], F. Calegari and D. Geraghty use the (conjectural) existence and local properties of Galois representations arising from torsion in the coherent or Betti cohomology of locally symmetric spaces to prove general modularity lifting results over an arbitrary number field.

For instance, when  $F = \mathbb{Q}$  a local-global compatibility theorem is proved in [CG12<sup>+</sup>] for Katz modular forms of weight 1 defined over an arbitrary  $\mathbb{Z}_p$ -algebra, and a minimal modularity lifting result is deduced using commutative algebra techniques (cf. Theorem 3.11 and Corollary 1.5 of *op.cit.*). Similarly, assume that  $g > 1$  and let  $\mathfrak{p} \mid p$  be a prime of  $F$ . If we chose a weight  $\kappa$  whose  $\mathfrak{p}$ -components are all equal to 1, it is natural to conjecture that the corresponding Galois representations arising from torsion in cohomology are unramified at  $\mathfrak{p}$ . But one major problem in approaching this conjecture is the lack of a good  $T_{\mathfrak{p}}$  Hecke-operator acting on such torsion classes. (This problem is solved when  $F = \mathbb{Q}$ , thanks to [Gr90, Proposition 4.1]).

In this paper, we prove the conjecture on unramifiedness of Galois representations when  $g = 2$  (and  $p$  is inert in  $F$ , for simplicity), and in some other instances for general  $g$ . More precisely, we first construct a natural Hecke operator at  $\mathfrak{p} \mid p$  acting on the cohomology of  $\mathcal{M}_{\mathcal{O}/\varpi^m}$  in any degree, and then we use this and the techniques of patching complexes of Calegari-Geraghty ([CG12<sup>+</sup>, 6]) to prove:

**Theorem 1.1.** *Assume that  $p > 3$  is inert in the quadratic totally real field  $F$ , and that  $\rho$  is a Frobenius-distinguished Galois representation arising from  $H^\bullet(\mathcal{M}_{\mathcal{O}/\varpi^m}^{\text{tor}}, \omega^{\mathbf{1}}(-\mathbf{D}))$  for some  $m \leq \infty$ . If  $\rho \otimes \overline{\mathbb{F}}_p$  is absolutely irreducible and has large image then the representation  $\rho$  is unramified at  $p$ .*

(Here  $\mathcal{M}^{\text{tor}}$  is a toroidal compactification of  $\mathcal{M}$ , and  $\mathbf{D}$  its boundary divisor). For convenience, we stated here a weaker result than what is proved in section 4 of the paper: cf. Proposition 5.5, Theorem 5.15, and Theorem 5.18.

**1.2. Sketch of the construction of the Hecke operator  $T_{\mathfrak{p}}$  on torsion classes.** Let  $\mathfrak{p}$  be a prime of  $F$  above  $p$  with residual degree  $f_{\mathfrak{p}}$  and inertial degree  $e_{\mathfrak{p}}$ . The naïve attempt of defining

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<sup>1</sup>If  $p$  is possibly ramified in  $F$  we should enhance the objects classified by the moduli problem with suitable linear algebra data, to obtain the Pappas-Rapoport splitting model of  $\mathcal{M}$ , cf. [PR05]. For simplicity we omit this from the introduction. On the same line, we do not discuss here the quotient  $\text{Sh}$  of  $\mathcal{M}$  which should be considered when defining Hilbert modular classes, cf. 2.2.

$T_{\mathfrak{p}}f$  for a Hilbert modular form  $f \in H^0(\mathcal{M}_{\mathcal{O}/\varpi^m}^{\text{tor}}, \omega^{\kappa}(-D))$  via the “sum”

$$(T_{\mathfrak{p}}f)(A, \dots) := \frac{1}{\text{Nm}_{F/\mathbb{Q}}(\mathfrak{p})} \sum_C f(A/C, \dots)$$

over suitable subgroup schemes  $C \subset A[\mathfrak{p}]$  of rank  $p^{f_{\mathfrak{p}}}$  does not make sense when  $\varpi^m = 0$  over the base scheme. When the weight  $\kappa$  of  $f$  is large enough<sup>2</sup> this is not an issue as we can first lift  $f$  to characteristic zero, apply the obvious  $T_{\mathfrak{p}}$  operator there, and finally reduce modulo  $\varpi^m$ . The problem is that not all Hilbert modular forms (for instance, in  $H^0(\mathcal{M}_{\mathcal{O}/\varpi^m}^{\text{tor}}, \omega^1)$ ) lift to characteristic zero. It is not clear *a priori* why a Hecke operator  $T_{\mathfrak{p}}$  should act with the expected  $q$ -expansion on non-liftable forms. Moreover, for our purposes we would like to have a  $T_{\mathfrak{p}}$  Hecke-action on the entire cohomology  $H^{\bullet}(\mathcal{M}_{\mathcal{O}/\varpi^m}^{\text{tor}}, \omega^1)$ , and not just on  $H^0$ , so that the trick adopted in [Gr90, Proposition 4.1] when  $g = 1$  (which relies on lifting a mod  $p$  Katz elliptic modular form of weight 1 to an *a priori* meromorphic form in characteristic zero) does not help either.

Our construction is inspired by the work of B. Conrad [Con07]. Let us motivate by recalling the definition of the Hecke operator at a prime  $\mathfrak{q} \nmid pN$  of  $F$ . Normally, we construct a covering  $\mathcal{M}(\mathfrak{q})$  of  $\mathcal{M}$  by adding an Iwahori level structure at  $\mathfrak{q}$ , and consider the (Hecke) correspondence attached to the two canonical projections  $\pi_1, \pi_2 : \mathcal{M}(\mathfrak{q}) \rightarrow \mathcal{M}$  given respectively by forgetting the Iwahori level structure, and by quotienting by it. More precisely, the Hecke action of  $T_{\mathfrak{q}}$  is obtained by composing the natural map (twisted by the Kodaira-Spencer isomorphism, cf. 3.9)

$$(1.2.1) \quad \pi_{1*}\pi_2^*\omega^{\kappa} \rightarrow \pi_{1*}\pi_1^*\omega^{\kappa}$$

with the finite flat trace map

$$(1.2.2) \quad \text{tr}_{\text{ff}} : \pi_{1*}\pi_1^*\omega^{\kappa} \rightarrow \omega^{\kappa},$$

dividing by  $\text{Nm}_{F/\mathbb{Q}}(\mathfrak{q})$ , and finally taking the cohomology.

It is a non-obvious fact that this geometric approach to define the Hecke action works also in characteristic  $p$  when  $g = 1$  and  $\mathfrak{q} = (p)$ , as proved in [Con07, 4.5]. The reason being that the composition  $\pi_{1*}\pi_2^*\omega^k \rightarrow \omega^k$  of the morphisms (1.2.1) and (1.2.2) between sheaves on  $\mathcal{M}_{\mathcal{O}}$  has image inside  $p \cdot \omega^k$  if  $k \geq 1$ , because it is so over the open dense ordinary locus.

Unfortunately, when  $g > 1$  there are additional complications as the projections  $\pi_1, \pi_2$  are not finite flat anymore over the special fiber of  $\mathcal{M}$ . For instance, when  $g = 2$  and  $p$  is inert in  $F$ , the preimage under  $\pi_1$  (or  $\pi_2$ ) of the superspecial points of the special fiber of  $\mathcal{M}$  is the union of two projective lines. To overcome this issue, we use in this paper the dualizing trace map

$$\text{tr} : R\pi_{1*}\mathcal{D}_{\mathfrak{p}} \rightarrow \mathcal{D}$$

instead of the finite flat trace map to construct the Hecke operator. (Here  $\mathcal{D}_{\mathfrak{p}}$  and  $\mathcal{D}$  denote the dualizing complexes on  $\mathcal{M}(\mathfrak{p})$  and on  $\mathcal{M}$  respectively). Twisting by the Kodaira-Spencer map we can identify  $\mathcal{D}$  with the automorphic line bundle  $\omega^2$  on  $\mathcal{M}$ , and the compatibility between the dualizing trace map and the finite flat trace map guarantee that our construction of  $T_{\mathfrak{p}}$  coincides with the usual one on the ordinary locus of  $\mathcal{M}$ . Following [Con07], it suffices to show that the composition  $R\pi_{1*}\pi_2^*\omega^{\kappa} \rightarrow \omega^{\kappa}$  factors through  $\text{Nm}_{F/\mathbb{Q}}(\mathfrak{p}) \cdot \omega^{\kappa}$  in the derived category of bounded complexes of coherent modules of  $\mathcal{M}$ . It is not difficult to check this over the ordinary locus (cf. Proposition 3.16).

Now we come to the second technical issue, which was pointed out to us by Pilloni.<sup>3</sup> Unlike in the case of  $F = \mathbb{Q}$ , since we are working with the derived category, knowing the factorization over the ordinary locus *a priori* does **not** imply the factorization over the entire  $\mathcal{M}$ . Luckily, when  $\kappa$

<sup>2</sup>Rigorously speaking, we need  $\omega^{\kappa}$  to be ample, which is stronger than just requiring  $\kappa$  to be large. See [TX13<sup>+</sup>, Theorem 1.9] for a conjectural ampleness condition.

<sup>3</sup>This was a gap in an early version of this paper. We are very grateful to Pilloni for communicating this to us.

satisfies certain natural and mild conditions (cf (3.10.1)), this implication holds. Indeed, by some standard homological algebra argument (cf Proposition 3.17), the factorization question is reduced to proving that the (set-theoretical) support of  $R^j \pi_{1*} \pi_2^* \omega_{\mathbb{F}_p}^\kappa$  (for  $j > 0$ ) has codimension at least  $j + 1$  in the special fiber  $\mathcal{M}_{\mathbb{F}_p}$ , which is the content of Proposition 3.18.

Let us point out that the proof of Proposition 3.18 makes use of some deep geometric result of the map  $\pi_1 : \mathcal{M}(\mathfrak{p})_{\mathbb{F}_p} \rightarrow \mathcal{M}_{\mathbb{F}_p}$ , along the line of [GKa12, He12, TX13<sup>+</sup>]. We single out this proof in Section 4 as we believe that it has its own interest. To summarize the key points: we show that

- all fibers of  $\pi_{1,\mathbb{F}_p}$  are unions of products of  $\mathbb{P}^1$ -bundles, and
- the restriction of  $\pi_2^* \omega^\kappa$  to each relevant  $\mathbb{P}^1$ -bundle (meaning those affect the proof of Proposition 3.18) is  $\mathcal{O}(n)$  for some  $n > -1$ , and hence the higher derived pushforward vanishes.

**1.3. Sketch of the proof of Theorem 1.1.** Having available a Hecke operator at  $\mathfrak{p}$  acting on torsion classes, we can prove (cf. Proposition 5.5) that representations arising from  $H^0(\mathcal{M}_{\mathcal{O}/\varpi^m}^{\text{tor}}, \omega^1(-D))$  are unramified at  $p$  (we assume for simplicity that  $p$  is inert in  $F$ , also we only look at Frobenius-distinguished representations). As in the elliptic modular case (cf. [Ed92] and [CG12<sup>+</sup>, pf. Theorem 3.11]) the proof of this fact uses that the Hecke operator  $T_{\mathfrak{p}}$  has the expected  $q$ -expansion on  $H^0$  (Remark 3.13), and consists basically in computing the commutator between  $T_{\mathfrak{p}}$  and the total Hasse invariant acting on forms of weight  $(1, \dots, 1)$ . By Serre duality, the result on  $H^0$  can also be read as an unramifiedness result for representations arising from  $H^g$ .

The cohomology groups  $H^*(\mathcal{M}_{\mathcal{O}/\varpi^m}^{\text{tor}}, \omega^1(-D))$  are modules for the Hecke algebra  $\mathbb{T}$  generated by the Hecke operators attached to primes away from  $p\mathcal{N}$ , together with the diamond operators (cf. 5.1). After localizing at a maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}$  attached to some mod  $p$  irreducible Galois representation  $\bar{\rho} : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$  which is modular of weight  $\mathbf{1}$ , the cohomology becomes a module for the full universal framed  $\mathcal{O}$ -deformation ring  $R_p$  of  $\bar{\rho}|_{G_p}$  (with fixed determinant). This ring contains a *proper* ideal  $\mathcal{I}$  cutting out the locus of unramified lifts. The existence of the  $T_{\mathfrak{p}}$  operator implies that the action of  $R_p$  on  $H^0$  and  $H^g$  factors via  $R_p/\mathcal{I}$ . Moreover, when  $g = 2$ , a “squeezing” argument (cf. Proposition 5.12) gives:

$$(1.3.1) \quad \mathcal{I}^4 \cdot H^1(\mathcal{M}_{\mathcal{O}/\varpi^m}^{\text{tor}}, \omega^1(-D))_{\mathfrak{m}} = 0.$$

At this point, to proceed in the argument, we need to introduce some extra ramification at sets of Taylor-Wiles primes. Denote by  $r$  the number of places of  $F$  dividing the level  $\mathcal{N}$  of Sh. For each  $N \geq 1$  let  $Q_N$  be a set of cardinality  $q \geq 1$  of Taylor-Wiles primes congruent to 1 modulo  $p^N$ . By adding level structure at  $Q_N$  to the moduli space  $\mathcal{M}$ , one can construct as in [CG12<sup>+</sup>, 6.1] complexes computing the cohomology of  $\mathcal{M}(Q_N)_{\mathcal{O}/\varpi^m}$  localized at a suitable maximal ideal  $\mathfrak{m}_{Q_N}$ . Such complexes are endowed with an action of a Hecke algebra  $\mathbb{T}_{Q_N, \mathfrak{m}_{Q_N}}$  and of the group-algebra  $\mathcal{O}[(\mathbb{Z}/p^N\mathbb{Z})^q]$ . Following [CG12<sup>+</sup>, pf. Theorem 6.3], we can patch these complexes together when  $N$  increases. Introducing framing by  $j := 4(r+1) - 1$  variables, this process produces a perfect complex  $P_{\infty}^{\square,*}$  of  $S_{\infty}^{\square} := \mathcal{O}[[x_1, \dots, x_{q+j}]]$ -modules. Choosing  $q$  to be the dimension of a suitable dual Selmer group, we see that  $H^{\bullet}(P_{\infty}^{\square,*})$  is endowed with an action of a complete local Noetherian  $R_p$ -algebra  $R'_{\infty}$  of relative Krull dimension  $4r + q - [F : \mathbb{Q}]$  (cf. 5.13). Since  $g = 2$  and  $\mathcal{I}^4 \cdot H^* = 0$  by (1.3.1), we conclude that the action of  $R'_{\infty}$  on the patched modules factors via  $R_{\infty} := R'_{\infty}/\mathcal{I}^4$ .

The fact that

$$\dim(R_p/\mathcal{I}^n) = \dim(R_p/\mathcal{I}) = 4$$

implies that the following numeric condition is satisfied:

$$\dim R_{\infty} = \dim S_{\infty}^{\square} - [F : \mathbb{Q}].$$

This guarantees that

$$\text{codim}_{S_{\infty}^{\square}} H^*(P_{\infty}^{\square,*}) = [F : \mathbb{Q}].$$

Since  $[F : \mathbb{Q}]$  is also the length of our complexes computing coherent cohomology, Lemma 6.2 of [CG12<sup>+</sup>] implies that the patched complex  $P_\infty^{\square,*}$  is a *projective* resolution of its top-degree cohomology. In particular we have an isomorphism of  $R_p$ -modules:

$$\mathrm{Tor}_i^{S_\infty^\square}(H^g(P_\infty^{\square,*}), \mathcal{O}/\varpi^m) \simeq H^i(\mathcal{M}_{\mathcal{O}/\varpi^m}^{\mathrm{tor}}, \omega^1(-D))_{\mathfrak{m}}.$$

Loosely speaking, the coherent cohomology of  $\mathcal{M}_{\mathcal{O}/\varpi^m}$  is controlled by the top-degree cohomology of some patched complex  $P_\infty^{\square,*}$ . Since  $H^g(P_\infty^{\square,*})$  is built by patching the top cohomologies of the schemes  $\mathcal{M}(Q_N)_{\mathcal{O}/\varpi^m}$  for varying  $N \geq 1$ , and since these give rise to unramified representations, we deduce Theorem 1.1 (cf. Theorem 5.15, and Theorem 5.18).

**1.4. Final comments.** In Theorem 1.1, the assumption that  $p$  is inert in  $F$  is just meant to keep our arguments as simple as possible. In particular, the case in which  $p$  is ramified can be treated by working over the Pappas-Rapoport splitting model of the Deligne-Pappas modular schemes. Also, it seems to be possible that, using similar techniques and some results on the global geometry of the Goren-Oort strata of Hilbert modular varieties from [TX13<sup>+</sup>], one can prove analogs of Theorem 1.1 for an arbitrary  $F$  assuming that for each prime  $\mathfrak{p} \mid p$  in  $F$  we have  $e_{\mathfrak{p}} f_{\mathfrak{p}} \leq 2$ . Moreover, the assumption of Frobenius distinguishness of  $\bar{\rho}$  (cf. 5.7) can probably be removed by suitable adaptation of the doubling arguments of [CG12<sup>+</sup>, 3.6-7] and [Wi14].

We remark that the only point in which we used  $g = 2$  in the argument is to show that a 4-dimensional quotient of the local deformation ring  $R_p$  was acting on our cohomology (namely,  $R_p/\mathcal{I}^4$ ). The argument just described would go through exactly in the same way for an arbitrary degree  $g = [F : \mathbb{Q}]$  and arbitrary  $p > 3$  if one could prove the following conjecture (cf. Conjecture 5.17):

**Conjecture.** *Let  $F$  be a totally real number field of degree  $g$  and denote by  $\mathcal{I}$  the ideal cutting out the unramified locus of the deformation ring  $R_p$ . There exists a positive integer  $n$  depending on  $g$  such that:*

$$\mathcal{I}^n \cdot H^\bullet(\mathcal{M}_{\mathcal{O}/\varpi^m}^{\mathrm{tor}}, \omega^1(-D))_{\mathfrak{m}} = 0.$$

**1.5. Organization of the paper.** In Section 2 we recall the definition of the Hilbert Shimura varieties, and the stratification induced on them by generalized partial Hasse invariants. Since we allow  $p$  to ramify in  $F$ , we work with the Pappas-Rapoport splitting model  $\mathcal{M}^{\mathrm{PR}}$ . In Section 3 we construct the Hecke operator  $T_{\mathfrak{p}}$  acting on the cohomology of  $\mathcal{M}_{\mathcal{O}/\varpi^m}^{\mathrm{PR}}$  ( $m \leq \infty$ ) with coefficient in an automorphic sheaf of paritious weight  $\kappa$  (satisfying some conditions), assuming Proposition 3.18, which is subsequently proved in Section 4. In Section 5 we use the construction of  $T_{\mathfrak{p}}$  operator to prove unramifiedness of the representations arising from (non-liftable) Katz Hilbert modular *forms* of weight **1**, and then we specialize to the case  $g = 2$  to prove unramifiedness of Hilbert modular *classes*.

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## 2. SPLITTING MODELS OF HILBERT MODULAR SCHEMES

We present here some preliminaries on splitting models of Hilbert modular schemes, as constructed by G. Pappas and M. Rapoport in [PR05]; we follow [RX14<sup>+</sup>]. We also recall the stratification induced by suitable generalized partial Hasse invariants on the special fiber of such models (cf. [RX14<sup>+</sup>]).

**2.1. Setup.** Let  $\overline{\mathbb{Q}}$  denote the algebraic closure of  $\mathbb{Q}$  inside  $\mathbb{C}$ . We fix a rational prime  $p$  and a field isomorphism  $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$ . Base changes of algebras and schemes will often be denoted by a subscript, if no confusion arises.

Let  $F$  be a totally real field of degree  $g > 1$ , with ring of integers  $\mathcal{O}_F$  and group of totally positive units  $\mathcal{O}_F^{\times,+}$ . Denote by  $\mathfrak{d} := \mathfrak{d}_F$  the different ideal of  $F/\mathbb{Q}$ . Let  $\mathfrak{C} := \{\mathfrak{c}_1, \dots, \mathfrak{c}_h\}$  be a fixed set of representatives for the elements of the narrow class group of  $F$ , chosen to be coprime to  $p$ .

We fix a large enough coefficient field  $E$  which is a finite Galois extension of  $\mathbb{Q}_p$  inside  $\overline{\mathbb{Q}}_p$ . We require that  $E$  contains the images of all  $p$ -adic embeddings of  $F(\sqrt{u}; u \in \mathcal{O}_F^{\times,+})$  into  $\overline{\mathbb{Q}}_p$ . Let  $\mathcal{O}$  denote the valuation ring of  $E$ ; choose a uniformizer  $\varpi$  of  $\mathcal{O}$  and denote by  $\mathbb{F}$  the residue field.

We write the prime ideal factorization of  $p\mathcal{O}_F$  as  $\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ , where  $r$  and  $e_i$  are positive integers. We also set  $\mathbb{F}_{\mathfrak{p}_i} = \mathcal{O}_F/\mathfrak{p}_i$  and  $f_i = [\mathbb{F}_{\mathfrak{p}_i} : \mathbb{F}_p]$ . Let  $\overline{\mathbb{F}}_p$  denote the residue field of  $\overline{\mathbb{Z}}_p$ , and let  $\sigma$  denote the arithmetic Frobenius on  $\overline{\mathbb{F}}_p$ . We label the embeddings of  $\mathbb{F}_{\mathfrak{p}_i}$  into  $\overline{\mathbb{F}}_p$  (or, equivalently, into  $\mathbb{F}$ ) as  $\{\tau_{\mathfrak{p}_i,1}, \dots, \tau_{\mathfrak{p}_i,f_i}\}$  so that  $\sigma \circ \tau_{\mathfrak{p}_i,j} = \tau_{\mathfrak{p}_i,j+1}$  for all  $j$ , with the convention that  $\tau_{\mathfrak{p}_i,j}$  stands for  $\tau_{\mathfrak{p}_i,j \pmod{f_i}}$ . For each  $\mathfrak{p}_i$ , denote by  $F_{\mathfrak{p}_i}$  the completion of  $F$  for the  $\mathfrak{p}_i$ -adic topology. Let  $W(\mathbb{F}_{\mathfrak{p}_i})$  denote the ring of integers of the maximal subfield of  $F_{\mathfrak{p}_i}$  unramified over  $\mathbb{Q}_p$ . The residue field of  $W(\mathbb{F}_{\mathfrak{p}_i})$  is identified with  $\mathbb{F}_{\mathfrak{p}_i}$ . Each embedding  $\tau_{\mathfrak{p}_i,j} : \mathbb{F}_{\mathfrak{p}_i} \rightarrow \mathbb{F}$  of residue fields induces an embedding  $W(\mathbb{F}_{\mathfrak{p}_i}) \rightarrow \mathcal{O}$  which we denote by the same symbol.

Let  $\Sigma$  denote the set of embeddings of  $F$  into  $\overline{\mathbb{Q}}$ , which is further identified with the set of embeddings of  $F$  into  $\mathbb{C}$  or  $\overline{\mathbb{Q}}_p$  or  $E$ . Let  $\Sigma_{\mathfrak{p}_i}$  denote the subset of  $\Sigma$  consisting of all the  $p$ -adic embeddings of  $F$  inducing the  $p$ -adic place  $\mathfrak{p}_i$ . For each  $i$  and each  $j = 1, \dots, f_i$ , there are exactly  $e_i$  elements in  $\Sigma_{\mathfrak{p}_i}$  that induce the embedding  $\tau_{\mathfrak{p}_i,j} : W(\mathbb{F}_{\mathfrak{p}_i}) \rightarrow \mathcal{O}$ ; we label these elements as  $\tau_{\mathfrak{p}_i,j}^1, \dots, \tau_{\mathfrak{p}_i,j}^{e_i}$ . *There is no canonical choice of such labelling, but we fix one for the rest of this paper.*

We choose a uniformizer  $\varpi_i$  for the ring of integers  $\mathcal{O}_{F_{\mathfrak{p}_i}}$  of  $F_{\mathfrak{p}_i}$ . Let  $E_{\mathfrak{p}_i}(x)$  denote the minimal polynomial of  $\varpi_i$  over the ring  $W(\mathbb{F}_{\mathfrak{p}_i})$ : it is an Eisenstein polynomial. Using the embedding  $\tau_{\mathfrak{p}_i,j}$ , we can view this polynomial as an element  $E_{\mathfrak{p}_i,j}(x) := \tau_{\mathfrak{p}_i,j}(E_{\mathfrak{p}_i}(x))$  of  $\mathcal{O}[x]$ . We have:

$$E_{\mathfrak{p}_i,j}(x) = (x - \tau_{\mathfrak{p}_i,j}^1(\varpi_i)) \cdots (x - \tau_{\mathfrak{p}_i,j}^{e_i}(\varpi_i)).$$

**2.2. Pappas-Rapoport splitting models.** Let  $S$  be a locally Noetherian  $\mathcal{O}$ -scheme. A *Hilbert-Blumenthal abelian  $S$ -scheme* (HBAS) with real multiplication by  $\mathcal{O}_F$  is the datum of an abelian  $S$ -scheme  $A$  of relative dimension  $g$ , together with a ring embedding  $\mathcal{O}_F \rightarrow \text{End}_S A$ . We have natural direct sum decompositions

$$\omega_{A/S} = \bigoplus_{i=1}^r \omega_{A/S, \mathfrak{p}_i} = \bigoplus_{i=1}^r \bigoplus_{j=1}^{f_i} \omega_{A/S, \mathfrak{p}_i, j},$$

where  $W(\mathbb{F}_{\mathfrak{p}_i}) \subseteq \mathcal{O}_{F_{\mathfrak{p}_i}}$  acts on  $\omega_{A/S, \mathfrak{p}_i, j}$  via  $\tau_{\mathfrak{p}_i, j}$ .

Let  $\mathfrak{c} \in \mathfrak{C}$  be a fractional ideal of  $F$  (coprime to  $p$ ), with cone of positive elements  $\mathfrak{c}^+$ . We say a HBAS  $A$  over  $S$  is  $\mathfrak{c}$ -polarized if there is an  $S$ -isomorphism  $\lambda : A^\vee \rightarrow A \otimes_{\mathcal{O}_F} \mathfrak{c}$  of HBAS's under which the symmetric elements (resp. the polarizations) of  $\text{Hom}_{\mathcal{O}_F}(A, A^\vee)$  correspond to the elements of  $\mathfrak{c}$  (resp.  $\mathfrak{c}^+$ ) in  $\text{Hom}_{\mathcal{O}_F}(A, A \otimes_{\mathcal{O}_F} \mathfrak{c})$ . For such a HBAS, each  $\omega_{A/S, \mathfrak{p}_i, j}$  is a locally free sheaf over  $S$  of rank  $e_i$ .

Let  $\mathcal{N}$  be a non-zero proper ideal of  $\mathcal{O}_F$  coprime to  $p$ . A  $\Gamma_{00}(\mathcal{N})$ -level structure on a HBAS  $A$  over  $S$  is an  $\mathcal{O}_F$ -equivariant closed embedding of  $S$ -schemes  $i : \mu_{\mathcal{N}} \rightarrow A$ , where  $\mu_{\mathcal{N}} := (\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathbb{G}_m)[\mathcal{N}]$  is the Cartier dual of the constant  $S$ -group scheme  $\mathcal{O}_F/\mathcal{N}$ .

Fix a non-zero proper ideal  $\mathcal{N}$  of  $\mathcal{O}_F$  coprime to  $p$ , and assume that  $\mathcal{N}$  does not divide neither  $2\mathcal{O}_F$  nor  $3\mathcal{O}_F$ . Denote by  $\underline{\mathcal{M}}_{\mathfrak{c}}^{\text{PR}} = \underline{\mathcal{M}}_{\mathfrak{c}, \mathcal{N}}^{\text{PR}}$  the functor that assigns to a locally Noetherian  $\mathcal{O}$ -scheme  $S$  the set of isomorphism classes of tuples  $(A, \lambda, i, \underline{\mathcal{F}})$ , where:

- $(A, \lambda)$  is a  $\mathfrak{c}$ -polarized HBAS over  $S$  with real multiplication by  $\mathcal{O}_F$ ,
- $i$  is a  $\Gamma_{00}(\mathcal{N})$ -level structure,
- $\underline{\mathcal{F}}$  is a collection  $(\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)})_{i=1, \dots, r; j=1, \dots, f_i; l=0, \dots, e_i}$  of locally free sheaves over  $S$  such that
  - $0 = \mathcal{F}_{\mathfrak{p}_{i,j}}^{(0)} \subsetneq \mathcal{F}_{\mathfrak{p}_{i,j}}^{(1)} \subsetneq \dots \subsetneq \mathcal{F}_{\mathfrak{p}_{i,j}}^{(e_i)} = \omega_{A/S, \mathfrak{p}_{i,j}}$  and each  $\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)}$  is stable under the  $\mathcal{O}_F$ -action (not just the action of  $W(\mathbb{F}_{\mathfrak{p}_i})$ ),
  - each subquotient  $\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)}/\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)}$  is a locally free  $\mathcal{O}_S$ -module of rank one (and hence  $\text{rank}_{\mathcal{O}_S} \mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)} = l$ ), and
  - the action of  $\mathcal{O}_F$  on each subquotient  $\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)}/\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)}$  factors through  $\mathcal{O}_F \xrightarrow{\tau_{\mathfrak{p}_{i,j}}^l} \mathcal{O}$ , or equivalently, this subquotient is annihilated by  $[\varpi_i] - \tau_{\mathfrak{p}_{i,j}}^l(\varpi_i)$ , where  $[\varpi_i]$  denotes the action of  $\varpi_i$  as an element of  $\mathcal{O}_{F_{\mathfrak{p}_i}}$ .

We use  $\underline{\mathcal{M}}_{\mathfrak{c}}^{\text{DP}}$  to denote the functor obtained from  $\underline{\mathcal{M}}_{\mathfrak{c}}^{\text{PR}}$  by forgetting the filtrations  $\underline{\mathcal{F}}$ .

Both  $\underline{\mathcal{M}}_{\mathfrak{c}}^{\text{PR}}$  and  $\underline{\mathcal{M}}_{\mathfrak{c}}^{\text{DP}}$  carry an action of  $\mathcal{O}_F^{\times,+}$ :

$$(2.2.1) \quad \text{for } u \in \mathcal{O}_F^{\times,+}, \quad \langle u \rangle : (A, \lambda, i, \underline{\mathcal{F}}) \mapsto (A, u\lambda, i, \underline{\mathcal{F}}).$$

This action is trivial on the subgroup  $(\mathcal{O}_{F, \mathcal{N}}^{\times})^2$  of  $\mathcal{O}_F^{\times,+}$ , where  $\mathcal{O}_{F, \mathcal{N}}^{\times}$  denotes the group of units that are congruent to 1 modulo  $\mathcal{N}$ .

It is well known (cf. Deligne-Pappas [DP94], Pappas-Rapoport [PR05], and Sasaki [Sa14<sup>+</sup>]) that the functor  $\underline{\mathcal{M}}_{\mathfrak{c}}^{\text{DP}}$  (resp.  $\underline{\mathcal{M}}_{\mathfrak{c}}^{\text{PR}}$ ) is represented by an  $\mathcal{O}$ -scheme of finite type, that we denote  $\mathcal{M}_{\mathfrak{c}}^{\text{DP}}$  (resp.  $\mathcal{M}_{\mathfrak{c}}^{\text{PR}}$ ). Moreover, the moduli space  $\mathcal{M}_{\mathfrak{c}}^{\text{DP}}$  is normal ([DP94]). Let  $\mathcal{M}_{\mathfrak{c}}^{\text{Ra}}$  denote its smooth locus, called the *Rapoport locus* ([Ra78]). Then  $\mathcal{M}_{\mathfrak{c}}^{\text{Ra}}$  is the open subscheme of  $\mathcal{M}_{\mathfrak{c}}^{\text{DP}}$  parameterizing those HBAS for which the cotangent space at the origin  $\omega_{A/S}$  is a locally free  $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S)$ -module of rank one. The natural morphism

$$\pi : \mathcal{M}_{\mathfrak{c}}^{\text{PR}} \rightarrow \mathcal{M}_{\mathfrak{c}}^{\text{DP}}$$

is projective, and it induces an isomorphism of an open subscheme of  $\mathcal{M}_{\mathfrak{c}}^{\text{PR}}$  onto  $\mathcal{M}_{\mathfrak{c}}^{\text{Ra}}$ . As in [RX14<sup>+</sup>, Theorem 2.9], one can prove that  $\mathcal{M}_{\mathfrak{c}}^{\text{PR}}$  is smooth over  $\mathcal{O}$ .

If  $\mathcal{N}$  is sufficiently divisible, the actions of  $\mathcal{O}_F^{\times,+}/(\mathcal{O}_{F, \mathcal{N}}^{\times})^2$  on  $\mathcal{M}_{\mathfrak{c}}^{\text{PR}}$  and  $\mathcal{M}_{\mathfrak{c}}^{\text{DP}}$  are free on geometric points (cf. [ERX13<sup>+</sup>, §2.1]). In particular, the corresponding quotients  $\text{Sh}_{\mathfrak{c}}^{\text{PR}}$  and  $\text{Sh}_{\mathfrak{c}}^{\text{DP}}$  are  $\mathcal{O}$ -schemes of finite type, and the quotient morphisms are étale.

For  $? \in \{\text{DP}, \text{PR}, \text{Ra}\}$  we denote by  $\mathcal{A}_{\mathfrak{c}}^?$  the universal abelian scheme over  $\mathcal{M}_{\mathfrak{c}}^?$ . We set

$$\mathcal{M}^? := \coprod_{\mathfrak{c} \in \mathfrak{C}} \mathcal{M}_{\mathfrak{c}}^?, \quad \text{Sh}^? := \coprod_{\mathfrak{c} \in \mathfrak{C}} \text{Sh}_{\mathfrak{c}}^?, \quad \text{and} \quad \mathcal{A}^? := \coprod_{\mathfrak{c} \in \mathfrak{C}} \mathcal{A}_{\mathfrak{c}}^?.$$

Notice that the universal abelian scheme  $\mathcal{A}^?$  may not descent to  $\text{Sh}^?$ .

Denote by  $\omega_{\mathcal{A}^?/\mathcal{M}^?}$  the pull-back via the zero section of the sheaf of relative differentials of  $\mathcal{A}^?$  over  $\mathcal{M}^?$ . We let  $\underline{\mathcal{F}} = (\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)})$  denote the universal filtration of  $\omega_{\mathcal{A}^{\text{PR}}/\mathcal{M}^{\text{PR}}}$ . For each  $p$ -adic embedding

$\tau = \tau_{\mathfrak{p}_{i,j}}^l$  of  $F$  into  $\overline{\mathbb{Q}}_p$ , we set

$$\dot{\omega}_\tau := \mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)} / \mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)};^4$$

it is an automorphic line bundle on the splitting model  $\mathcal{M}^{\text{PR}}$ . We provide  $\dot{\omega}_\tau$  with an action of  $\mathcal{O}_F^{\times,+}$  following [DT04]: a positive unit  $u \in \mathcal{O}_F^{\times,+}$  maps a local section  $s$  of  $\dot{\omega}_\tau$  to  $u^{-1/2} \cdot \langle u \rangle^*(s)$ , where  $\langle u \rangle$  is defined by (2.2.1). It is clear that this action factors through  $\mathcal{O}_F^{\times,+} / (\mathcal{O}_{F,\mathcal{N}}^{\times})^2$ .

Similarly, for each  $p$ -adic embedding  $\tau$  of  $F$ , we define

$$\begin{aligned} \dot{\varepsilon}_\tau &:= (\wedge_{\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{M}^{\text{PR}}}}^2 \mathcal{H}_{\text{dR}}^1(\mathcal{A}^{\text{PR}} / \mathcal{M}^{\text{PR}})) \otimes_{\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{M}^{\text{PR}}, \tau \otimes 1}} \mathcal{O}_{\mathcal{M}^{\text{PR}}} \\ &\cong (\mathfrak{c} \mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{M}^{\text{PR}}}) \otimes_{\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{M}^{\text{PR}}, \tau \otimes 1}} \mathcal{O}_{\mathcal{M}^{\text{PR}}}, \end{aligned}$$

where the canonical isomorphism is induced by the universal polarization as in [RX14<sup>+</sup>, Lemma 2.5]. While  $\dot{\varepsilon}_\tau$  is a trivial line bundle, it carries a non-trivial action of  $\mathcal{O}_F^{\times,+} / (\mathcal{O}_{F,\mathcal{N}}^{\times})^2$  given as follows: a positive unit  $u \in \mathcal{O}_F^{\times,+}$  maps a local section  $s$  of  $\dot{\varepsilon}_\tau$  to  $u^{-1} \cdot \langle u \rangle^*(s)$ .

It is proved in [RX14<sup>+</sup>, Theorem 2.9] that the sheaf of relative differentials  $\Omega_{\mathcal{M}^{\text{PR}}/\mathcal{O}}^1$  admits a canonical Kodaira-Spencer filtration whose successive subquotients are given by

$$\dot{\omega}_\tau^{\otimes 2} \otimes_{\mathcal{O}_{\mathcal{M}^{\text{PR}}}} \dot{\varepsilon}_\tau^{\otimes -1} \quad \text{for } \tau \in \Sigma.$$

**2.3. Toroidal compactifications.** For any ideal class  $\mathfrak{c} \in \mathfrak{C}$  fix a rational polyhedral admissible cone decomposition  $\Phi_{\mathfrak{c}}$  for each isomorphism class of  $\Gamma_{00}(\mathcal{N})$ -cusps of the  $\mathcal{O}$ -scheme  $\mathcal{M}_{\mathfrak{c}}^{\text{Ra}}$  ([DT04, 5.1]).

Denote by  $\mathcal{M}_{\mathfrak{c}}^{\text{PR}, \text{tor}}$  (resp.  $\mathcal{M}_{\mathfrak{c}}^{\text{DP}, \text{tor}}$ ) the corresponding toroidal compactification of  $\mathcal{M}_{\mathfrak{c}}^{\text{PR}}$  (resp.  $\mathcal{M}_{\mathfrak{c}}^{\text{DP}}$ ) as in [RX14<sup>+</sup>, 2.11]. The scheme  $\mathcal{M}_{\mathfrak{c}}^{\text{PR}, \text{tor}}$  is proper and smooth over  $\text{Spec } \mathcal{O}$ . We set  $\mathcal{M}^{?, \text{tor}} := \coprod_{\mathfrak{c} \in \mathfrak{C}} \mathcal{M}_{\mathfrak{c}}^{?, \text{tor}}$  for  $? \in \{\text{DP}, \text{PR}, \text{Ra}\}$ . The boundary  $\dot{\mathcal{D}} := \mathcal{M}^{?, \text{tor}} - \mathcal{M}^?$  is a relative simple normal crossing divisor on  $\mathcal{M}^{?, \text{tor}}$ .

Let  $\text{Sh}^{?, \text{tor}}$  denote the quotient of  $\mathcal{M}^{?, \text{tor}}$  by the action of the group  $\mathcal{O}_F^{\times,+} / (\mathcal{O}_{F,\mathcal{N}}^{\times})^2$ . Put  $\mathcal{D} := \text{Sh}^{?, \text{tor}} - \text{Sh}^?$ ; it is the quotient of  $\dot{\mathcal{D}}$  and it is a divisor with simple normal crossings.

For any  $\tau \in \Sigma$ , there are automorphic line bundles  $\dot{\omega}_\tau^{\text{tor}}$  and  $\dot{\varepsilon}_\tau^{\text{tor}}$  over  $\mathcal{M}^{\text{PR}, \text{tor}}$  constructed as in *loc.cit.*. To lighten the load on notation, we will simply write  $\dot{\omega}_\tau$  and  $\dot{\varepsilon}_\tau$  to denote these sheaves when no confusion arises. Both these line bundles carry natural actions of  $\mathcal{O}_F^{\times,+} / (\mathcal{O}_{F,\mathcal{N}}^{\times})^2$  as described earlier, and they descend to line bundles over  $\text{Sh}^{\text{PR}, \text{tor}}$ , which we denote by  $\omega_\tau$  and  $\varepsilon_\tau$  respectively. We warn the reader that  $\varepsilon_\tau$  may not be the trivial line bundle over  $\text{Sh}^{\text{PR}, \text{tor}}$ .

**2.4. Geometric Hilbert modular forms.** A *paritious weight*  $\kappa$  is a tuple  $((k_\tau)_{\tau \in \Sigma}, w) \in \mathbb{Z}^\Sigma \times \mathbb{Z}$  such that  $k_\tau \equiv w \pmod{2}$  for every  $\tau \in \Sigma$ . We say that  $\kappa$  is *regular* if moreover  $k_\tau > 1$  for all  $\tau \in \Sigma$ .

For  $\kappa = ((k_\tau)_{\tau \in \Sigma}, w)$  a paritious weight, we define

$$\dot{\omega}^\kappa := \bigotimes_{\tau \in \Sigma} (\dot{\omega}_\tau^{\otimes k_\tau} \otimes_{\mathcal{O}_{\mathcal{M}^{\text{PR}, \text{tor}}}} \dot{\varepsilon}_\tau^{\otimes (w - k_\tau)/2}), \text{ and } \omega^\kappa := \bigotimes_{\tau \in \Sigma} (\omega_\tau^{\otimes k_\tau} \otimes_{\mathcal{O}_{\text{Sh}^{\text{PR}, \text{tor}}}} \varepsilon_\tau^{\otimes (w - k_\tau)/2}).$$

They are line bundles over  $\mathcal{M}^{\text{PR}, \text{tor}}$  and  $\text{Sh}^{\text{PR}, \text{tor}}$ , respectively. We remind the reader that these line bundles depend on the fixed choice of an ordering of the  $\tau_{\mathfrak{p}_{i,j}}^l$ 's.

A (*geometric*) *Hilbert modular form* over a Noetherian  $\mathcal{O}$ -algebra  $R$ , of level  $\Gamma_{00}(\mathcal{N})$  and (paritious) weight  $\kappa$  is an element of the finite  $R$ -module  $H^0(\text{Sh}_R^{\text{PR}, \text{tor}}, \omega_R^\kappa)$ , where the subscript  $R$

<sup>4</sup>The additional dot in the notation  $\dot{\omega}_\tau$  is placed in order to distinguish this sheaf from its descent  $\omega_\tau$  to  $\text{Sh}^{\text{PR}}$ , which will be introduced later.



indicates base change to  $R$  over  $\mathcal{O}$ . Such a form is called *cuspidal* if it belongs to the submodule  $H^0(\mathrm{Sh}_R^{\mathrm{PR},\mathrm{tor}}, \omega_R^\kappa(-D))$ . By the K ocher principle ([DT04, Th eor em 7.1]), if  $[F : \mathbb{Q}] > 1$ , we have

$$H^0(\mathcal{M}_R^{\mathrm{PR},\mathrm{tor}}, \dot{\omega}_R^\kappa) = H^0(\mathcal{M}_R^{\mathrm{PR}}, \dot{\omega}_R^\kappa), \text{ and hence } H^0(\mathrm{Sh}_R^{\mathrm{PR},\mathrm{tor}}, \omega_R^\kappa) = H^0(\mathrm{Sh}_R^{\mathrm{PR}}, \omega_R^\kappa).$$

Let  $\mathbf{S}$  denote the a finite set of places of  $F$  containing the place dividing  $p\mathcal{N}$  and the archimedean places. The polynomial ring

$$\mathbb{T}_{\mathbf{S}}^{\mathrm{univ}} := \mathcal{O}[t_{\mathfrak{q}}; \mathfrak{q} \text{ a place of } F \text{ not in } \mathbf{S}]$$

is called the *universal tame Hecke algebra*. It acts on the cohomology groups  $H^j(\mathrm{Sh}^{\mathrm{PR},\mathrm{tor}}, \omega^\kappa)$  and  $H^j(\mathrm{Sh}^{\mathrm{PR},\mathrm{tor}}, \omega^\kappa(-D))$  via the assignment  $t_{\mathfrak{q}} \mapsto T_{\mathfrak{q}}$ , where  $T_{\mathfrak{q}}$  denotes the (tame) Hecke operator at  $\mathfrak{q}$  as constructed in [RX14<sup>+</sup>, 2.14] and acting on the appropriate cohomology group.

**Notation 2.5.** For the rest of this paper, if no confusion arises, we will drop the superscript PR appearing in the schemes introduced in the previous subsections. In particular, we set  $\mathcal{A} := \mathcal{A}^{\mathrm{PR}}$ ,  $\mathcal{M} := \mathcal{M}^{\mathrm{PR}}$ , and  $\mathrm{Sh} := \mathrm{Sh}^{\mathrm{PR}}$ . These are schemes over  $\mathcal{O}$ , and we denote their special fibers by  $\mathcal{A}_{\mathbb{F}}$ ,  $\mathcal{M}_{\mathbb{F}}$ , and  $\mathrm{Sh}_{\mathbb{F}}$  respectively. Analogous notations are used to denote toroidal compactifications.

**2.6. Generalized partial Hasse invariants on splitting models.** We recall some constructions from [RX14<sup>+</sup>, §3]. For each  $\tau = \tau_{\mathfrak{p},j}^l$  with  $l \neq 1$ , multiplication by  $\varpi_i$  induces a well defined morphism:

$$\begin{aligned} m_{\varpi_i,j}^{(l)} : \mathcal{F}_{\mathfrak{p},j}^{(l)} / \mathcal{F}_{\mathfrak{p},j}^{(l-1)} &\longrightarrow \mathcal{F}_{\mathfrak{p},j}^{(l-1)} / \mathcal{F}_{\mathfrak{p},j}^{(l-2)} \\ z &\longmapsto [\varpi_i](\tilde{z}), \end{aligned}$$

where  $\tilde{z}$  is a lift to  $\mathcal{F}_{\mathfrak{p},j}^{(l)}$  of the local section  $z$ . Hence  $m_{\varpi_i,j}^{(l)}$  induces a section

$$\dot{h}_\tau \in H^0(\mathcal{M}_{\mathbb{F}}, \dot{\omega}_{\tau_{\mathfrak{p},j}^l}^{\otimes -1} \otimes \dot{\omega}_{\tau_{\mathfrak{p},j}^{l-1}})$$

that is invariant under the action of  $\mathcal{O}_F^{\times,+} / (\mathcal{O}_{F,\mathcal{N}}^{\times})^2$ , and therefore descend to a section  $h_\tau \in H^0(\mathrm{Sh}_{\mathbb{F}}, \omega_{\tau_{\mathfrak{p},j}^l}^{\otimes -1} \otimes \omega_{\tau_{\mathfrak{p},j}^{l-1}})$ .

For  $\tau = \tau_{\mathfrak{p},j}^1$ , a morphism

$$\mathrm{Hasse}_{\varpi_i,j} : \dot{\omega}_{\tau_{\mathfrak{p},j}^1} = \mathcal{F}_{\mathfrak{p},j}^{(1)} \longrightarrow \omega_{\mathcal{A}_{\mathbb{F}}/\mathcal{M}_{\mathbb{F}},\mathfrak{p},j-1}^{(p)} / (\mathcal{F}_{\mathfrak{p},j-1}^{(e_i-1)})^{(p)} \cong \dot{\omega}_{\tau_{\mathfrak{p},j-1}^{e_i}}^{\otimes p}$$

is constructed in [RX14<sup>+</sup>, 3.1] as follows: let  $z$  be a local section of  $\dot{\omega}_{\tau_{\mathfrak{p},j}^1}$ ; since it is annihilated by  $[\varpi_i]$  acting on  $\mathcal{H}_{\mathrm{dR}}^1(\mathcal{A}_{\mathbb{F}}/\mathcal{M}_{\mathbb{F}})_{\mathfrak{p},j}$ ,  $z$  belongs to  $[\varpi_i]^{e_i-1} \cdot \mathcal{H}_{\mathrm{dR}}^1(\mathcal{A}_{\mathbb{F}}/\mathcal{M}_{\mathbb{F}})_{\mathfrak{p},j}$ . Write  $z = [\varpi_i]^{e_i-1} z'$  for a local section  $z'$  of  $\mathcal{H}_{\mathrm{dR}}^1(\mathcal{A}_{\mathbb{F}}/\mathcal{M}_{\mathbb{F}})_{\mathfrak{p},j}$  (note this  $z'$  belongs to  $\mathcal{H}_{\mathrm{dR}}^1$  but it might not belong to  $\omega$ ). We define  $\mathrm{Hasse}_{\varpi_i,j}(z)$  to be the image of  $V_{\mathfrak{p},j}(z')$  in  $\omega_{\mathcal{A}_{\mathbb{F}}/\mathcal{M}_{\mathbb{F}},\mathfrak{p},j-1}^{(p)} / (\mathcal{F}_{\mathfrak{p},j-1}^{(e_i-1)})^{(p)}$ . The homomorphism  $\mathrm{Hasse}_{\varpi_i,j}$  is well defined and it induces a section

$$\dot{h}_{\tau_{\mathfrak{p},j}^1} \in H^0(\mathcal{M}_{\mathbb{F}}, \dot{\omega}_{\tau_{\mathfrak{p},j}^1}^{\otimes -1} \otimes \dot{\omega}_{\tau_{\mathfrak{p},j-1}^{e_i}}^{\otimes p})$$

which descends to a section  $h_{\tau_{\mathfrak{p},j}^1} \in H^0(\mathrm{Sh}_{\mathbb{F}}, \omega_{\tau_{\mathfrak{p},j}^1}^{\otimes -1} \otimes \omega_{\tau_{\mathfrak{p},j-1}^{e_i}}^{\otimes p})$ .

**2.7. Goren-Oort stratification of  $\mathrm{Sh}_{\mathbb{F}}$ .** For each  $p$ -adic embedding  $\tau \in \Sigma$ , denote by  $X_\tau$  the zero locus of the generalized Hasse invariant  $h_\tau$  on  $\mathrm{Sh}_{\mathbb{F}}$ . In general, for a subset  $\mathbf{T} \subseteq \Sigma$ , we set  $X_{\mathbf{T}} := \cap_{\tau \in \mathbf{T}} X_\tau$ , with the convention that if  $\mathbf{T}$  is the empty set, this intersection is interpreted to be the entire space  $\mathrm{Sh}_{\mathbb{F}}$ . We define  $\dot{X}_\tau$  and  $\dot{X}_{\mathbf{T}}$  on  $\mathcal{M}_{\mathbb{F}}$  similarly. These  $X_{\mathbf{T}}$  (resp.  $\dot{X}_{\mathbf{T}}$ ) form the *Goren-Oort stratification* of  $\mathrm{Sh}_{\mathbb{F}}$  (resp.  $\mathcal{M}_{\mathbb{F}}$ ). We write

$$X_{\mathbf{T}}^\circ := X_{\mathbf{T}} \setminus \cup_{\mathbf{T}' \supsetneq \mathbf{T}} X_{\mathbf{T}'} \quad \text{and} \quad \dot{X}_{\mathbf{T}}^\circ := \dot{X}_{\mathbf{T}} \setminus \cup_{\mathbf{T}' \supsetneq \mathbf{T}} \dot{X}_{\mathbf{T}'}$$

for the corresponding open stratum.

We remark that, although the generalized partial Hasse-invariants  $h_\tau$  and  $\dot{h}_\tau$  depend on the choice of uniformizers  $\varpi_i$ , their zero loci  $X_\tau$  and  $\dot{X}_\tau$  do not. Moreover, each section  $\dot{h}_\tau$  extends to the fixed toroidal compactification  $\mathcal{M}_{\mathbb{F}}^{\text{tor}}$  of  $\mathcal{M}_{\mathbb{F}}$  and, by [RX14<sup>+</sup>, Theorem 3.10], each subscheme  $\dot{X}_\tau$  is disjoint from the cusps, which are ordinary points of the moduli space.

**Proposition 2.8.** *The following properties hold:*

- (1) *The closed subschemes  $X_\tau$  (resp.  $\dot{X}_\tau$ ) are proper and smooth divisors with simple normal crossings on  $\text{Sh}_{\mathbb{F}}$  (resp.  $\mathcal{M}_{\mathbb{F}}$ ). In particular,  $\dot{X}_{\mathbf{T}}$  and  $\dot{X}_{\mathbf{T}}$  for any proper subset  $\mathbf{T} \subset \Sigma$  are proper and smooth varieties over  $\mathbb{F}$ .*
- (2) *The union of the closed subschemes  $X_\tau$  (resp.  $\dot{X}_\tau$ ) when  $\tau$  varies over all embeddings of the form  $\tau_{\mathfrak{p},j}^l$  with  $l \neq 1$  coincides with the complement of  $\text{Sh}_{\mathbb{F}}^{\text{Ra}}$  (resp.  $\mathcal{M}_{\mathbb{F}}^{\text{Ra}}$ ) in  $\text{Sh}_{\mathbb{F}}$  (resp.  $\mathcal{M}_{\mathbb{F}}$ ).*
- (3) *The ordinary locus  $\text{Sh}_{\mathbb{F}}^{\text{ord}}$  (resp.  $\mathcal{M}_{\mathbb{F}}^{\text{ord}}$ ) of  $\text{Sh}_{\mathbb{F}}$  (resp.  $\mathcal{M}_{\mathbb{F}}$ ) coincides with the complement of the set  $\cup_{\tau \in \Sigma} X_\tau$  (resp.  $\cup_{\tau \in \Sigma} \dot{X}_\tau$ ).*

*Proof.* It is enough to prove the proposition over  $\mathcal{M}_{\mathbb{F}}$ . The first statement is contained in [RX14<sup>+</sup>, Theorem 3.10]. To prove the second statement we can work with closed points of  $\mathcal{M}_{\mathbb{F}}$ , since the Rapoport condition on the Lie algebra is an open condition. Let then  $(A, \lambda, i, \underline{\mathcal{F}})$  be a  $k$ -point of  $\mathcal{M}_{\mathbb{F}}$ , for  $k$  an extension field of  $\mathbb{F}$ . The abelian variety  $A$  with RM satisfies the Rapoport condition if and only if multiplication by  $\varpi_i$  induces an isomorphism  $\mathcal{F}_{\mathfrak{p},j}^{(l)} / \mathcal{F}_{\mathfrak{p},j}^{(l-1)} \rightarrow \mathcal{F}_{\mathfrak{p},j}^{(l-1)} / \mathcal{F}_{\mathfrak{p},j}^{(l-2)}$  for all  $i$ , all  $j$ , and all  $l \neq 1$ . This is clearly equivalent to the given condition on the divisors  $\dot{X}_\tau$ . Statement (3) follows combining (2) with the fact that every ordinary abelian variety with RM automatically satisfies the Rapoport condition ([AG05, Remark 3.6]), and the fact that the morphisms  $\text{Hase}_{\mathfrak{p},j}$  retrieve the usual partial Hasse invariants of [AG05] on the Rapoport locus.  $\square$

### 3. HECKE OPERATORS AT $p$ IN CHARACTERISTIC $p^m$

In this section, we construct the  $T_{\mathfrak{p}}$  operator acting on the cohomology of some automorphic line bundles with torsion coefficients. We work exclusively with the Pappas-Rapoport splitting model, as indicated in Notation 2.5.

We fix a prime ideal  $\mathfrak{p} \in \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ ; denote by  $f$  its residual degree, and by  $e$  its inertial degree. We write  $\varpi_{\mathfrak{p}}$  for the chosen uniformizer at  $\mathfrak{p}$  as in 2.1. For every fractional ideal  $\mathfrak{c} \in \mathfrak{C}$  we choose once and for all a positive isomorphism  $\theta_{\mathfrak{c}} : \mathfrak{c}\mathfrak{p} \simeq \mathfrak{c}'$  of  $\mathfrak{c}\mathfrak{p}$  with a (uniquely determined) fractional ideal  $\mathfrak{c}' \in \mathfrak{C}$ .

**3.1. Splitting models with Iwahori level structure.** For a fixed  $\mathfrak{c} \in \mathfrak{C}$  we define the corresponding splitting models with Iwahori level structure following [Pa95].

Let  $\underline{\mathcal{M}}_{\mathfrak{c}}(\mathfrak{p})$  denote the functor which assigns to a locally noetherian  $\mathcal{O}$ -scheme  $S$  the set of isomorphism classes of tuples  $((A, \lambda, i, \underline{\mathcal{F}}); (A', \lambda', i', \underline{\mathcal{F}}'); \phi)$  where:

- (1)  $[(A, \lambda, i, \underline{\mathcal{F}})]$  is an  $S$ -points of  $\mathcal{M}_{\mathfrak{c}}$ ,
- (2)  $[(A', \lambda', i', \underline{\mathcal{F}}')]$  is an  $S$ -points of  $\mathcal{M}_{\mathfrak{c}'}$ ,
- (3)  $\phi$  is an  $\mathcal{O}_F$ -equivariant  $S$ -isogeny  $A \rightarrow A'$  satisfying:
  - (a)  $\phi$  has degree  $p^f$ ,
  - (b)  $\phi$  is compatible with the polarizations, i.e.,  $\phi \circ \lambda \circ \phi^\vee = \tilde{\lambda}'$ , where  $\tilde{\lambda}'$  is the map  $(A')^\vee \rightarrow A' \otimes \mathfrak{c}$  induced by composing  $\lambda'$  with the map  $\mathfrak{c}' \xrightarrow{\theta_{\mathfrak{c}}^{-1}} \mathfrak{c}\mathfrak{p} \subset \mathfrak{c}$ ,
  - (c)  $\phi$  is compatible with the level structures, i.e.,  $\phi \circ i = i'$ ,

- (d) there is an  $\mathcal{O}_F$ -equivariant  $S$ -isogeny  $\psi : A' \rightarrow A \otimes \mathbf{c}(\mathbf{c}')^{-1}$  such that the compositions  $\psi \circ \phi$  and  $(\phi \otimes \mathbf{c}(\mathbf{c}')^{-1}) \circ \psi$  are the natural isogenies  $A \rightarrow A \otimes \mathbf{c}(\mathbf{c}')^{-1}$  and  $A' \rightarrow A' \otimes \mathbf{c}(\mathbf{c}')^{-1}$  induced by  $\mathcal{O}_F \subseteq \mathfrak{p}^{-1} \xrightarrow{\theta_{\mathbf{c}}} \mathbf{c}(\mathbf{c}')^{-1}$ , and
- (e)  $\phi$  and  $\psi$  are compatible with the filtrations, *i.e.*, for any indices  $i \in \{1, \dots, r\}$  and  $j \in \{1, \dots, f_i\}$  the morphisms of locally free  $\mathcal{O}_S$ -modules
- $$\phi^* : \omega_{A'/S, \mathfrak{p}_{i,j}} \rightarrow \omega_{A/S, \mathfrak{p}_{i,j}} \text{ and } \psi^* : \omega_{A/S, \mathfrak{p}_{i,j}} \cong \omega_{A/S, \mathfrak{p}_{i,j}} \otimes \mathbf{c}(\mathbf{c}')^{-1} \rightarrow \omega_{A'/S, \mathfrak{p}_{i,j}}$$
- preserve the filtrations  $\mathcal{F}_{\mathfrak{p}_{i,j}}^\bullet$  and  $\mathcal{F}_{\mathfrak{p}_{i,j}}^{\bullet'}$ .

We point out that the existence of  $\psi$  is equivalent to the fact that  $\ker(\phi) \subseteq A[\mathfrak{p}]$ . In particular, this means that, for  $\mathfrak{p}_i \neq \mathfrak{p}$ ,

$$\mathcal{F}_{\mathfrak{p}_{i,j}}'^{(l)} \xrightarrow{\phi^*} \mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)} \xrightarrow{\psi^*} \mathcal{F}_{\mathfrak{p}_{i,j}}'^{(l)}$$

are isomorphisms for all  $j = 1, \dots, f_i$  and  $l = 1, \dots, e_i$ . Moreover, the construction also gives a commutative diagram:

$$(3.1.1) \quad \begin{array}{ccc} A^\vee & \xrightarrow{\lambda} & A \otimes \mathbf{c} \\ \phi^\vee \uparrow & & \uparrow \psi \\ A'^\vee & \xrightarrow{\lambda'} & A' \otimes \mathbf{c}' \end{array}$$

We observe that the functor  $\underline{\mathcal{M}}_{\mathbf{c}}(\mathfrak{p})$  is canonically independent on the choice of  $\theta_{\mathbf{c}}$ . The natural transformation of functors  $\pi_1 : \underline{\mathcal{M}}_{\mathbf{c}}(\mathfrak{p}) \rightarrow \underline{\mathcal{M}}_{\mathbf{c}}$  induced by only keeping  $(A, \lambda, i, \mathcal{F})$  is also independent on  $\theta_{\mathbf{c}}$ , and relatively representable by [Pa95]. In particular,  $\underline{\mathcal{M}}_{\mathbf{c}}(\mathfrak{p})$  is represented by an  $\mathcal{O}$ -scheme  $\mathcal{M}_{\mathbf{c}}(\mathfrak{p})$  of finite type. The other natural transformation of functors  $\pi_2 = \pi_{2, \theta_{\mathbf{c}}} : \underline{\mathcal{M}}_{\mathbf{c}}(\mathfrak{p}) \rightarrow \underline{\mathcal{M}}_{\mathbf{c}}$  induced by only keeping  $(A', \lambda', i', \mathcal{F}')$  *does* depend on the choice of the isomorphism  $\theta_{\mathbf{c}}$  which we fixed at the beginning.

The group  $\mathcal{O}_F^{\times,+}/(\mathcal{O}_{F,\mathcal{N}}^\times)^2$  acts freely on  $\mathcal{M}_{\mathbf{c}}(\mathfrak{p})$  (by acting simultaneously on the polarizations of source and target of the isogenies). We denote by  $\text{Sh}_{\mathbf{c}}(\mathfrak{p})$  the corresponding quotient. We set  $\mathcal{M}(\mathfrak{p}) := \coprod_{\mathbf{c} \in \mathcal{C}} \mathcal{M}_{\mathbf{c}}(\mathfrak{p})$  and  $\text{Sh}(\mathfrak{p}) := \coprod_{\mathbf{c} \in \mathcal{C}} \text{Sh}_{\mathbf{c}}(\mathfrak{p})$ .

Both projective morphisms  $\pi_1$  and  $\pi_2$  are equivariant for the action of  $\mathcal{O}_F^{\times,+}/(\mathcal{O}_{F,\mathcal{N}}^\times)^2$  and induce maps:

$$\pi_1 : \text{Sh}(\mathfrak{p}) \rightarrow \text{Sh} \quad \text{and} \quad \pi_2 = \pi_{2, \{\theta_{\mathbf{c}}\}_{\mathbf{c}}} : \text{Sh}(\mathfrak{p}) \rightarrow \text{Sh}.$$

Although the morphism  $\pi_2$  depends on the auxiliary choice of  $\theta_{\mathbf{c}}$ 's, we point out that this dependence disappears as soon as we pass to the cohomology of an automorphic line bundle  $\omega^\kappa$  or  $\dot{\omega}^\kappa$  for a paritious weight  $\kappa$ . So the  $T_{\mathfrak{p}}$ -operator we consider later will be canonical.

**Remark 3.2.** When  $p$  is *unramified* in  $F$ , the moduli space  $\mathcal{M}_{\mathbf{c}}(\mathfrak{p})$  can also be described as the moduli space of tuples  $(A, \lambda, i; C)/S$  where:

- $[(A, \lambda, i)]$  is an  $S$ -point of  $\mathcal{M}_{\mathbf{c}}$  (the filtration datum on  $\omega_{A/S}$  is uniquely determined in this case);
- $C$  is a finite-flat, closed,  $\mathcal{O}_F$ -stable  $S$ -subgroup scheme of  $A[\mathfrak{p}]$  of rank  $p^f$  (which is isotropic with respect to the Weil paring induced by  $\lambda$ ).

It is not clear how such interpretation could be extended to the splitting model. For this reason, we defined  $\mathcal{M}_{\mathbf{c}}(\mathfrak{p})$  as a moduli space of isogenies.

**Proposition 3.3.** *The scheme  $\mathcal{M}_{\mathbf{c}}(\mathfrak{p})$  (resp.  $\text{Sh}_{\mathbf{c}}(\mathfrak{p})$ ) is a flat local complete intersection of relative dimension  $g$  over  $\mathcal{O}$ . In particular, it is Gorenstein, and hence Cohen-Macaulay. Moreover, the special fiber  $\mathcal{M}_{\mathbf{c}}(\mathfrak{p})_{\mathbb{F}}$  (resp.  $\text{Sh}_{\mathbf{c}}(\mathfrak{p})_{\mathbb{F}}$ ) is smooth outside a closed subscheme of codimension 1, and  $\mathcal{M}_{\mathbf{c}}(\mathfrak{p})$  (resp.  $\text{Sh}_{\mathbf{c}}(\mathfrak{p})$ ) is normal.*

*Proof.* It is enough to prove the statements for  $\mathcal{M}_c(\mathfrak{p})$ , since the natural quotient map  $\mathcal{M}_c(\mathfrak{p}) \rightarrow \mathrm{Sh}_c(\mathfrak{p})$  is finite and étale. The proposition then follows from a study of local models as in [DP94] and [Pa95] (cf. also [PR05] and [Sa14<sup>+</sup>]).

We begin by determining the deformation functor for points of  $\mathcal{M}_c(\mathfrak{p})$ . Our argument closely follows that in [RX14<sup>+</sup>, Theorem 2.9]. Let  $S_0 \hookrightarrow S$  be a closed immersion of locally noetherian  $\mathcal{O}$ -schemes whose ideal of definition  $\mathcal{I}$  satisfies  $\mathcal{I}^2 = 0$ . Consider an  $S_0$ -valued point  $x_0 = ((A, \lambda, i, \mathcal{F}), (A', \lambda', i', \mathcal{F}'); \phi)$  of  $\mathcal{M}_c(\mathfrak{p})$ . Let  $\mathcal{H}_{\mathrm{cris}}^1(A/S_0)$  denote the crystalline cohomology sheaf of  $A$  over  $S_0$ . The action of  $\mathcal{O}_F$  on  $A$  induces a natural direct sum decomposition:

$$\mathcal{H}_{\mathrm{cris}}^1(A/S_0) = \bigoplus_{i=1}^r \bigoplus_{j=1}^{f_i} \mathcal{H}_{\mathrm{cris}}^1(A/S_0)_{\mathfrak{p}_{i,j}},$$

where  $W(\mathbb{F}_{\mathfrak{p}_i}) \subseteq \mathcal{O}_{F_{\mathfrak{p}_i}}$  acts on  $\mathcal{H}_{\mathrm{cris}}^1(A/S_0)_{\mathfrak{p}_{i,j}}$  via  $\tau_{\mathfrak{p}_{i,j}}$ . Moreover  $\mathcal{H}_{\mathrm{cris}}^1(A/S_0)_{\mathfrak{p}_{i,j}}$  is a locally free module of rank two over

$$\mathcal{O}_{F_{\mathfrak{p}_i}} \otimes_{W(\mathbb{F}_{\mathfrak{p}_i}), \tau_{\mathfrak{p}_{i,j}}} \mathcal{O}_{S_0}^{\mathrm{cris}} \cong \mathcal{O}_{S_0}^{\mathrm{cris}}[x]/(E_{\mathfrak{p}_{i,j}}(x)).$$

Since  $S$  is a PD-thickening of  $S_0$ , we can evaluate the crystalline cohomology over  $S$  to obtain  $\mathcal{H}_{\mathrm{cris}}^1(A/S)_S$  and its direct summands  $\mathcal{H}_{\mathrm{cris}}^1(A/S)_{S, \mathfrak{p}_{i,j}}$ . The polarization  $\lambda : A^\vee \rightarrow A \otimes_{\mathcal{O}_F} \mathfrak{c}$  induces a non-degenerate, symplectic pairing

$$\langle \cdot, \cdot \rangle : \mathcal{H}_{\mathrm{cris}}^1(A/S_0)_{\mathfrak{p}_{i,j}} \times \mathcal{H}_{\mathrm{cris}}^1(A/S_0)_{\mathfrak{p}_{i,j}} \rightarrow \mathcal{O}_{S_0}^{\mathrm{cris}}, \text{ such that}$$

$$(3.3.1) \quad \langle ax, y \rangle = \langle x, ay \rangle \quad \text{and} \quad \langle ax, x \rangle = 0.$$

for  $a \in \mathcal{O}_{F_{\mathfrak{p}_i}}$  and  $x, y \in \mathcal{H}_{\mathrm{cris}}^1(A/S_0)_{\mathfrak{p}_{i,j}}$  (as proved in [RX14<sup>+</sup>, (2.9.1–2)]). For a subspace  $\mathcal{F}$  of  $\mathcal{H}_{\mathrm{cris}}^1(A/S_0)_{\mathfrak{p}_{i,j}}$ , we use  $\mathcal{F}^\perp$  to denote its orthogonal complement under the above pairing. The submodule  $\omega_{A/S_0, \mathfrak{p}_{i,j}}$  of  $\mathcal{H}_{\mathrm{cris}}^1(A/S_0)_{S_0, \mathfrak{p}_{i,j}}$  is (maximal) isotropic for this pairing. In particular,  $\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)} \subset (\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)})^\perp$  for all  $i, j, l$ .

Analogous constructions, notations, and properties can be introduced for the abelian scheme  $A'/S_0$ . The isogenies  $\phi$  and  $\psi$  define morphisms

$$\phi^* : \mathcal{H}_{\mathrm{cris}}^1(A'/S_0)_{\mathfrak{p}_{i,j}} \rightarrow \mathcal{H}_{\mathrm{cris}}^1(A/S_0)_{\mathfrak{p}_{i,j}} \quad \text{and} \quad \psi^* : \mathcal{H}_{\mathrm{cris}}^1(A/S_0)_{\mathfrak{p}_{i,j}} \rightarrow \mathcal{H}_{\mathrm{cris}}^1(A'/S_0)_{\mathfrak{p}_{i,j}}.$$

The commutative diagram (3.1.1) implies that

$$\langle x, \phi^*(y) \rangle = \langle \psi^*(x), y \rangle'$$

for  $x \in \mathcal{H}_{\mathrm{cris}}^1(A/S_0)_{\mathfrak{p}_{i,j}}$  and  $y \in \mathcal{H}_{\mathrm{cris}}^1(A'/S_0)_{\mathfrak{p}_{i,j}}$ . So it follows that

$$\phi^*(\mathcal{F}_{\mathfrak{p}_{i,j}}'^{(l)}) \subseteq \mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)}, \quad \phi^*((\mathcal{F}_{\mathfrak{p}_{i,j}}'^{(l)})^\perp) \subseteq (\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)})^\perp, \quad \psi^*(\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)}) \subseteq \mathcal{F}_{\mathfrak{p}_{i,j}}'^{(l)}, \quad \text{and} \quad \psi^*((\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)})^\perp) \subseteq (\mathcal{F}_{\mathfrak{p}_{i,j}}'^{(l)})^\perp.$$

Let

$$\mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)}(A/S_0) := \left\{ z \in (\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)})^\perp / \mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)} \mid [\varpi_i]z - \tau_{\mathfrak{p}_{i,j}}^l(\varpi_i)z = 0 \right\},$$

and similarly define  $\mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)}(A'/S_0)$  for each index  $i, j, l \geq 1$ ; so that we have natural morphisms

$$\phi_{\mathcal{H}}^* : \mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)}(A'/S_0) \rightarrow \mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)}(A/S_0) \quad \text{and} \quad \psi_{\mathcal{H}}^* : \mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)}(A/S_0) \rightarrow \mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)}(A'/S_0).$$

It is shown in the Claim of the proof of [RX14<sup>+</sup>, Theorem 2.9] that  $\mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)}(A/S_0)$  is a rank-two  $\mathcal{O}_{S_0}$ -subbundle of  $\mathcal{H}_{\mathrm{cris}}^1(A/S_0)_{S_0, \mathfrak{p}_{i,j}} / \mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)}$ . In particular, the sheaf  $\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)} / \mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)}$  is a rank-one  $\mathcal{O}_{S_0}$ -subbundle of  $\mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)}(A/S_0)$ .

By crystalline deformation theory for abelian schemes, lifting the  $S_0$ -point  $x_0$  associated to the isogeny  $\phi : A \rightarrow A'$  to  $S$  is equivalent to the following procedure, applied to each choice of  $i$  and  $j$ :

- (1) Write  $\mathcal{H}_{\mathfrak{p}_{i,j}}^{(1)}(A/S_0)_S$  and  $\mathcal{H}_{\mathfrak{p}_{i,j}}^{(1)}(A'/S_0)_S$  for the kernel of the map  $[\varpi_i] - \tau_{\mathfrak{p}_{i,j}}^1(\varpi_i)$  acting on  $\mathcal{H}_{\text{cris}}^1(A/S_0)_{S,\mathfrak{p}_{i,j}}$  and  $\mathcal{H}_{\text{cris}}^1(A'/S_0)_{S,\mathfrak{p}_{i,j}}$ , respectively.  
 Lift  $\mathcal{F}_{\mathfrak{p}_{i,j}}^{(1)} \subset \mathcal{H}_{\mathfrak{p}_{i,j}}^{(1)}(A/S_0)$  and  $\mathcal{F}'_{\mathfrak{p}_{i,j}} \subset \mathcal{H}_{\mathfrak{p}_{i,j}}^{(1)}(A'/S_0)$  to (isotropic)<sup>5</sup> rank-one  $\mathcal{O}_S$ -subbundles  $\tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(1)} \subset \mathcal{H}_{\mathfrak{p}_{i,j}}^{(1)}(A/S_0)_S$  and  $\tilde{\mathcal{F}}'_{\mathfrak{p}_{i,j}} \subset \mathcal{H}_{\mathfrak{p}_{i,j}}^{(1)}(A'/S_0)_S$  satisfying the conditions  $\phi^*(\tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(1)}) \subseteq \tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(1)}$  and  $\psi^*(\tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(1)}) \subseteq \tilde{\mathcal{F}}'_{\mathfrak{p}_{i,j}}^{(1)}$ .  
 (2) Once lifts  $\tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(l)}$  and  $\tilde{\mathcal{F}}'_{\mathfrak{p}_{i,j}}^{(l)}$  are chosen for all  $l \leq t-1$ , set

$$\mathcal{H}_{\mathfrak{p}_{i,j}}^{(t)}(A/S_0)_S := \left\{ z \in (\tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(t-1)})^\perp / \tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(t-1)} \mid [\varpi_i]z - \tau_{\mathfrak{p}_{i,j}}^t(\varpi_i)z = 0 \right\}$$

and similarly for  $\mathcal{H}_{\mathfrak{p}_{i,j}}^{(t)}(A'/S_0)_S$ ; by the proof of Claim (1) in [RX14<sup>+</sup>, Theorem 2.9], the sheaves  $\mathcal{H}_{\mathfrak{p}_{i,j}}^{(t)}(A/S_0)_S$  and  $\mathcal{H}_{\mathfrak{p}_{i,j}}^{(t)}(A'/S_0)_S$  are rank-two  $\mathcal{O}_S$ -subbundles of  $\mathcal{H}_{\text{cris}}^1(A/S_0)_{S,\mathfrak{p}_{i,j}} / \tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(t-1)}$  and  $\mathcal{H}_{\text{cris}}^1(A'/S_0)_{S,\mathfrak{p}_{i,j}} / \tilde{\mathcal{F}}'_{\mathfrak{p}_{i,j}}^{(t-1)}$ , respectively. Then we need to lift  $\mathcal{F}_{\mathfrak{p}_{i,j}}^{(t)} / \mathcal{F}_{\mathfrak{p}_{i,j}}^{(t-1)} \subset \mathcal{H}_{\mathfrak{p}_{i,j}}^{(t)}(A/S_0)$  and  $\mathcal{F}'_{\mathfrak{p}_{i,j}} / \mathcal{F}'_{\mathfrak{p}_{i,j}}^{(t-1)} \subset \mathcal{H}_{\mathfrak{p}_{i,j}}^{(t)}(A'/S_0)$  to (isotropic)<sup>6</sup> rank-one  $\mathcal{O}_S$ -subbundles  $\tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(t)/(t-1)}$  and  $\tilde{\mathcal{F}}'_{\mathfrak{p}_{i,j}}^{(t)/(t-1)}$  of  $\mathcal{H}_{\mathfrak{p}_{i,j}}^{(t)}(A/S_0)_S$  and  $\mathcal{H}_{\mathfrak{p}_{i,j}}^{(t)}(A'/S_0)_S$  respectively, such that

$$\phi^*(\tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(t)/(t-1)}) \subseteq \tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(t)/(t-1)} \quad \text{and} \quad \psi^*(\tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(t)/(t-1)}) \subseteq \tilde{\mathcal{F}}'_{\mathfrak{p}_{i,j}}^{(t)/(t-1)}.$$

After this, we define  $\tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(t)}$  to be the preimage of  $\tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(t)/(t-1)}$  under the natural projection  $(\tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(t-1)})^\perp \rightarrow (\tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(t-1)})^\perp / \tilde{\mathcal{F}}_{\mathfrak{p}_{i,j}}^{(t-1)}$  and define  $\tilde{\mathcal{F}}'_{\mathfrak{p}_{i,j}}^{(t)}$  similarly.

Following [Pa95, Lemma 3.3.2], we claim that for any locally noetherian  $\mathcal{O}$ -scheme  $S_0$ , the tuple

$$(\mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)}(A/S_0), \mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)}(A'/S_0), \phi_{\mathcal{H}}^*, \psi_{\mathcal{H}}^*)$$

is Zariski locally isomorphic (over  $S_0$ ) to the “constant” tuple obtained from  $(\mathcal{O}_{S_0}^2, \mathcal{O}_{S_0}^2, u_{\mathfrak{p}_{i,j}}^{(l)}, v_{\mathfrak{p}_{i,j}}^{(l)})$  by scalar extension, where:

$$u_{\mathfrak{p}_{i,j}}^{(l)} := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & \tau_{\mathfrak{p}_{i,j}}^l(\varpi_i) \end{pmatrix} & \text{if } \mathfrak{p}_i = \mathfrak{p} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \mathfrak{p}_i \neq \mathfrak{p} \end{cases} \quad \text{and} \quad v_{\mathfrak{p}_{i,j}}^{(l)} := \begin{cases} \begin{pmatrix} \tau_{\mathfrak{p}_{i,j}}^l(\varpi_i) & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \mathfrak{p}_i = \mathfrak{p} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \mathfrak{p}_i \neq \mathfrak{p}. \end{cases}$$

This is obvious at a closed point of  $S_0$  of characteristic zero. At a closed point  $s \in S_0$  of characteristic  $p$ , we observe that  $\psi_{\mathcal{H}}^* \circ \phi_{\mathcal{H}}^*$  and  $\phi_{\mathcal{H}}^* \circ \psi_{\mathcal{H}}^*$  are both zero; and the kernels of  $\phi_{\mathcal{H}}^*$  and  $\psi_{\mathcal{H}}^*$  are at most one-dimensional, as the kernels of  $\phi^*$  and  $\psi^*$  on  $H_{\text{dR}}^1(A'/k(s))_{\mathfrak{p}_{i,j}}$  and  $H_{\text{dR}}^1(A/k(s))_{\mathfrak{p}_{i,j}}$  respectively are one-dimensional. It follows that the images of  $\psi_{\mathcal{H}}^*$  and  $\phi_{\mathcal{H}}^*$  are both one-dimensional. The claim is clear from that.

For each  $i, j, l$  let  $G_{i,j}^{(l)}$  denote the functor defined on the category of locally noetherian  $\mathcal{O}$ -schemes which associates to a scheme  $S_0$  the set of isomorphism classes of pairs  $(\mathcal{F}, \mathcal{F}')$  of invertible  $\mathcal{O}_{S_0}$ -subbundles of  $\mathcal{O}_{S_0}^2$  satisfying  $u_{\mathfrak{p}_{i,j}}^{(l)}(\mathcal{F}') \subseteq \mathcal{F}$  and  $v_{\mathfrak{p}_{i,j}}^{(l)}(\mathcal{F}) \subseteq \mathcal{F}'$ . The functor  $G_{i,j}^{(l)}$  is represented by an  $\mathcal{O}$ -scheme  $\mathfrak{N}_{i,j}^{(l)}$  which is a closed subscheme of a fiber product of two Grassmannians. Denote by  $\mathfrak{N}$  the fiber product of  $\mathcal{O}$ -schemes  $\prod_{i,j,l} \mathfrak{N}_{i,j}^{(l)}$ .

Let  $x_0$  be a closed point of  $\mathcal{M}_c(\mathfrak{p})$  with finite residue field, and let  $\mathcal{U}$  be an open neighborhood of  $x_0$ . Modulo shrinking  $\mathcal{U}$  to an open subscheme, and after choosing rigidifications as above, we obtain a morphism  $\pi : \mathcal{U} \rightarrow \mathfrak{N}$ . This induces a map of formal schemes  $\pi_{x_0} : \mathcal{M}_{x_0} \rightarrow \mathcal{N}_{x_0}$ , where

<sup>5</sup>This isotropic condition is automatic by condition (3.3.1); see the proof of [RX14<sup>+</sup>, Theorem 2.9] for more details.

<sup>6</sup>Once again, this isotropic condition is automatic by condition (3.3.1).

$\mathcal{M}_{x_0}$  denotes the formal completion of  $\mathcal{M}_c(\mathfrak{p})$  at  $x_0$ , and  $\mathcal{N}_{x_0}$  denotes the finite étale covering of the formal completion of  $\mathfrak{N}$  at  $\pi(x_0)$  having special fiber  $\{x_0\}$ . Using the description (1) and (2) of the deformation functor of  $x_0$ , we conclude that  $\pi_{x_0}$  induces an isomorphism at the level of the Zariski tangent spaces at  $x_0$  of the special fibers of  $\mathcal{M}_{x_0}$  and  $\mathcal{N}_{x_0}$ . Now the argument at the end of [DP94, §3] implies that  $\pi_{x_0}$  is an isomorphism of formal schemes, and therefore  $\pi$  is étale at  $x_0$ .

It now suffices to study the local geometry of  $\mathfrak{N}_{i,j}^{(l)}$ . But this is well-known: when  $\mathfrak{p}_i \neq \mathfrak{p}$ ,  $\mathfrak{N}_{i,j}^{(l)}$  is isomorphic to  $\mathbb{P}_{\mathcal{O}}^1$ ; when  $\mathfrak{p}_i = \mathfrak{p}$ , over a small enough open subspace  $\mathfrak{U} \subseteq \mathfrak{N}_{i,j}^{(l)}$  where both  $\mathcal{F}$  and  $\mathcal{F}'$  are trivialized, we can define a map

$$h : \mathfrak{U} \rightarrow \mathcal{O}[U_{\mathfrak{p},j}^{(l)}, V_{\mathfrak{p},j}^{(l)}] / (U_{\mathfrak{p},j}^{(l)} V_{\mathfrak{p},j}^{(l)} - \tau_{\mathfrak{p},j}^l(\varpi_i))$$

such that the maps

$$u_{\mathfrak{p},j}^{(l)} : \mathcal{F}' \rightarrow \mathcal{F} \quad \text{and} \quad v_{\mathfrak{p},j}^{(l)} : \mathcal{F} \rightarrow \mathcal{F}'$$

are given by multiplication by  $U_{\mathfrak{p},j}^{(l)}$  and  $V_{\mathfrak{p},j}^{(l)}$ , respectively. It is easy to prove that this map  $h$  is étale (e.g. the case  $N = 1$  of [Pa95, Proposition 4.2.2]). So to sum up,  $\mathfrak{N}$  and hence  $\mathcal{M}_c(\mathfrak{p})$  is étale locally isomorphic to the spectrum of the following ring:

$$\mathcal{O}[W_{\mathfrak{p},j}^{(l)}]_{\mathfrak{p}_i \neq \mathfrak{p}, j=1, \dots, f_i, l=1, \dots, e_i} \otimes \mathcal{O} \left( \bigotimes_{j=1}^f \bigotimes_{l=1}^e \frac{\mathcal{O}[U_{\mathfrak{p},j}^{(l)}, V_{\mathfrak{p},j}^{(l)}]}{(U_{\mathfrak{p},j}^{(l)} V_{\mathfrak{p},j}^{(l)} - \tau_{\mathfrak{p},j}^l(\varpi_{\mathfrak{p}}))} \right),$$

where  $\varpi_{\mathfrak{p}}$  is the uniformizer at  $\mathfrak{p}$  (as set up at the beginning of Section 3).  $\square$

**Notation 3.4.** The proof of the above Proposition suggests us to define a stratification of  $\mathcal{M}(\mathfrak{p})_{\mathbb{F}}$ , following [He12, GK12]. For two subsets  $\mathbf{S}, \mathbf{S}' \subseteq \Sigma_{\mathfrak{p}}$  such that  $\mathbf{S} \cup \mathbf{S}' = \Sigma_{\mathfrak{p}}$ , we write  $Y_{\mathbf{S}, \mathbf{S}'}$  for the subscheme of  $\mathcal{M}(\mathfrak{p})_{\mathbb{F}}$  where

- $\phi^* : \mathcal{F}_{\mathfrak{p}_i,j}^{(l)} / \mathcal{F}_{\mathfrak{p}_i,j}^{(l-1)} \rightarrow \mathcal{F}_{\mathfrak{p}_i,j}^{(l)} / \mathcal{F}_{\mathfrak{p}_i,j}^{(l-1)}$  vanishes if  $\tau_{\mathfrak{p}_i,j}^l \in \mathbf{S}$ , and
- $\psi^* : \mathcal{F}_{\mathfrak{p}_i,j}^{(l)} / \mathcal{F}_{\mathfrak{p}_i,j}^{(l-1)} \rightarrow \mathcal{F}_{\mathfrak{p}_i,j}^{(l)} / \mathcal{F}_{\mathfrak{p}_i,j}^{(l-1)}$  vanishes if  $\tau_{\mathfrak{p}_i,j}^l \in \mathbf{S}'$ .

Note that  $\phi^* \circ \psi^*$  and  $\psi^* \circ \phi^*$  vanishes modulo  $p$  by the moduli problem; so we need the condition  $\mathbf{S} \cup \mathbf{S}' = \Sigma_{\mathfrak{p}}$  otherwise  $Y_{\mathbf{S}, \mathbf{S}'}$  is nonempty.

By Corollary 3.5 below, the open strata are given by

$$Y_{\mathbf{S}, \mathbf{S}'}^{\circ} := Y_{\mathbf{S}, \mathbf{S}'} \setminus \left( \bigcup_{(\mathbf{S}_1, \mathbf{S}'_1) \supsetneq (\mathbf{S}, \mathbf{S}')} Y_{\mathbf{S}_1, \mathbf{S}'_1} \right),$$

where  $(\mathbf{S}_1, \mathbf{S}'_1) \supsetneq (\mathbf{S}, \mathbf{S}')$  means  $\mathbf{S}_1 \supsetneq \mathbf{S}$  and  $\mathbf{S}'_1 \supsetneq \mathbf{S}'$ , and the equalities of sets cannot hold simultaneously.

**Corollary 3.5.** (1) *There are  $3^{\#\Sigma_{\mathfrak{p}}}$  closed strata  $Y_{\mathbf{S}, \mathbf{S}'}$ .*

(2) *Each  $Y_{\mathbf{S}, \mathbf{S}'}$  is a smooth variety over  $\mathbb{F}$  of dimension*

$$g - (\#\mathbf{S} + \#\mathbf{S}' - \#\Sigma_{\mathfrak{p}}).$$

(3) *For another pair  $\mathbf{S}_1, \mathbf{S}'_1 \subseteq \Sigma_{\mathfrak{p}}$  such that  $\mathbf{S}_1 \cup \mathbf{S}'_1 = \Sigma_{\mathfrak{p}}$ , we have*

$$Y_{\mathbf{S}, \mathbf{S}'} \cap Y_{\mathbf{S}_1, \mathbf{S}'_1} = Y_{\mathbf{S} \cup \mathbf{S}_1, \mathbf{S}' \cup \mathbf{S}'_1}.$$

(4) *The irreducible components of  $\mathcal{M}(\mathfrak{p})_{\mathbb{F}}$  are exactly those given by  $Y_{\mathbf{S}, \Sigma_{\mathfrak{p}} \setminus \mathbf{S}}$  for  $\mathbf{S}$  a subset of  $\Sigma_{\mathfrak{p}}$ .*

*Proof.* (1) is because each  $\tau \in \Sigma_{\mathfrak{p}}$  (independently) can belong to one of  $\mathbf{S}$  or  $\mathbf{S}'$ , or belong to both.

(2) It follows from the local model computation in the proof of Proposition 3.3 that  $Y_{\mathbf{S}, \mathbf{S}'}$  is étale locally isomorphic to the spectrum of the following ring

$$\mathbb{F}[W_{\mathbf{p}, j}^{(l)}]_{\mathbf{p}_i \neq \mathbf{p}, j=1, \dots, f_i, l=1, \dots, e_i} \otimes_{\mathbb{F}} \bigotimes_{\tau_{\mathbf{p}, j}^l \in \Sigma_p} \begin{cases} \mathbb{F}[U_{\mathbf{p}, j}^{(l)}, V_{\mathbf{p}, j}^{(l)}]/(U_{\mathbf{p}, j}^{(l)}) & \text{if } \tau_{\mathbf{p}, j}^l \in \mathbf{S}, \\ \mathbb{F}[U_{\mathbf{p}, j}^{(l)}, V_{\mathbf{p}, j}^{(l)}]/(V_{\mathbf{p}, j}^{(l)}) & \text{if } \tau_{\mathbf{p}, j}^l \in \mathbf{S}', \\ \mathbb{F}[U_{\mathbf{p}, j}^{(l)}, V_{\mathbf{p}, j}^{(l)}]/(U_{\mathbf{p}, j}^{(l)}, V_{\mathbf{p}, j}^{(l)}) & \text{if } \tau_{\mathbf{p}, j}^l \in \mathbf{S} \cap \mathbf{S}'. \end{cases}$$

It follows that  $Y_{\mathbf{S}, \mathbf{S}'}$  is smooth of the said dimension. The last statement is an immediate consequence of the dimension formula.

(3) follows from definition.

(4) is an immediate corollary of (2) and (3).  $\square$

**Remark 3.6.** It seems to be possible to give a global description of these strata  $Y_{\mathbf{S}, \mathbf{S}'}$  using the techniques of [He12, TX13<sup>+</sup>]. In Section 4, some partial information of  $Y_{\mathbf{S}, \mathbf{S}'}$  is used to prove the key technical Proposition 3.18.

We have the following (well-known) result:

**Proposition 3.7.** *The morphisms of  $\mathcal{O}$ -schemes  $\pi_1, \pi_2 : \text{Sh}(\mathbf{p}) \rightarrow \text{Sh}$  are finite and flat over the ordinary locus of  $\text{Sh}(\mathbf{p})$ .*

*Proof.* It is enough to prove the statement for the analogous morphisms between fine moduli spaces. For a locally noetherian  $\mathcal{O}$ -scheme  $S$ , we say that an  $S$ -point  $\phi : A_1 \rightarrow A_2$  of  $\mathcal{M}_{\mathbf{c}}(\mathbf{p})$  is ordinary if and only if  $A_1$  (or, equivalently,  $A_2$ ) is an ordinary abelian scheme.

Recall that we have fixed a positive isomorphism  $\theta_{\mathbf{c}} : \mathbf{c}\mathbf{p} \simeq \mathbf{c}'$ , with  $\mathbf{c}' \in \mathfrak{C}$ . We need to prove that, restricting to the ordinary locus,  $\pi_1 : \mathcal{M}_{\mathbf{c}}(\mathbf{p})^{\text{ord}} \rightarrow \mathcal{M}_{\mathbf{c}}^{\text{ord}}$  and  $\pi_2 : \mathcal{M}_{\mathbf{c}}(\mathbf{p})^{\text{ord}} \rightarrow \mathcal{M}_{\mathbf{c}'}^{\text{ord}}$  are finite and flat.

By [AG05, Remark 3.6] an ordinary abelian variety with RM automatically satisfies the Rapoport condition, and as such it admits exactly one filtration satisfying the Pappas-Rapoport conditions of section 2.2. In what follows we can then forget about the filtrations appearing in the tuples classified by  $\mathcal{M}_{\mathbf{c}}(\mathbf{p})^{\text{ord}}$ ,  $\mathcal{M}_{\mathbf{c}}^{\text{ord}}$ , and  $\mathcal{M}_{\mathbf{c}'}^{\text{ord}}$ . The proof of the proposition is now analogous to the one of [DR73, V, Lemme 1.12]. For brevity, we only show the finite flatness of:

$$\pi_1 : \mathcal{M}_{\mathbf{c}}(\mathbf{p})^{\text{ord}} \rightarrow \mathcal{M}_{\mathbf{c}}^{\text{ord}}.$$

Let  $k$  be an  $\mathcal{O}$ -algebra which is an algebraically closed field of characteristic  $p$ , and let  $x : \text{Spec } k \rightarrow \mathcal{M}_{\mathbf{c}}^{\text{ord}}$  be a  $k$ -point of  $\mathcal{M}_{\mathbf{c}}^{\text{ord}}$  defining a tuple  $(A, \lambda, i)$ . (Recall that we can forget about the filtrations). The fiber  $T$  of  $\pi_1$  over  $x$  is the  $k$ -scheme representing the functor which assigns to a locally noetherian  $k$ -scheme  $S$  the set of isomorphism classes of finite-flat, closed,  $\mathcal{O}_F$ -stable,  $\lambda$ -isotropic,  $S$ -subgroup schemes  $C \subset (A \times_k S)[\mathbf{p}]$  of rank  $p^f$ .

Since  $A$  is ordinary, the connected-étale exact sequence of  $A[\mathbf{p}]$  is of the form:

$$(3.7.1) \quad 0 \rightarrow \mathbb{F}_{\mathbf{p}} \otimes_{\mathbb{Z}} \mu_{p/k} \rightarrow A[\mathbf{p}] \rightarrow \mathbb{F}_{\mathbf{p}} \otimes_{\mathbb{Z}} \left( \frac{\mathbb{Z}}{p\mathbb{Z}} \right)_k \rightarrow 0,$$

where the morphisms are equivariant for the natural action of  $\mathcal{O}_F/\mathbf{p} = \mathbb{F}_{\mathbf{p}}$ . If  $C \subset (A \times_k S)[\mathbf{p}]$  represents an  $S$ -point of  $T$ , we can write  $S = S' \coprod S''$  where  $C \times_S S'$  is equal to the connected part  $(A \times_k S')[\mathbf{p}]^{\circ} \simeq \mathbb{F}_{\mathbf{p}} \otimes_{\mathbb{Z}} \mu_{p/S'}$  of  $(A \times_k S')[\mathbf{p}]$ , and  $C \times_S S''$  is isomorphic to  $\mathbb{F}_{\mathbf{p}} \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})_{S''}$ . (To see this notice that if  $y : \text{Spec } \bar{l} \rightarrow S$  is a closed point of  $S$  for some field extension  $\bar{l}$  of  $k$ , and if the group of geometric points  $C_y(\bar{l})$  of the fiber of  $C$  at  $y$  is non-trivial, then the existence of an action of  $\mathbb{F}_{\mathbf{p}}$  on  $C_y(\bar{l})$  forces  $C_y$  to be isomorphic to  $\mathbb{F}_{\mathbf{p}} \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})_{/l}$ . On the other hand, if  $C_y$  has trivial étale quotient, then it is contained in the connected component of the identity of  $(A \times_k l)[\mathbf{p}]$ ; the existence of the  $\mathbb{F}_{\mathbf{p}}$ -action then forces this inclusion to be an equality).

In correspondence of the decomposition  $S = S' \coprod S''$  we can write  $T = T' \coprod T''$  where  $T'$  is the reduced  $k$ -scheme that assigns  $(A \times_k S)[\mathbf{p}]^{\circ}$  to any locally noetherian  $k$ -scheme  $S$ , while  $T''$

represents the functor of  $\mathbb{F}_p$ -equivariant splittings of the exact sequence (3.7.1). Since  $T''$  is a torsor under the group-scheme  $\mathrm{Hom}_{\mathbb{F}_p \otimes_{\mathbb{Z}} k}(A[\mathfrak{p}]^{\mathrm{ét}}, A[\mathfrak{p}]^{\circ}) \simeq \mathbb{F}_p \otimes_{\mathbb{Z}} \mu_{p/k}$ , we see that  $T \simeq \mathrm{Spec} k \coprod (\mathbb{F}_p \otimes_{\mathbb{Z}} \mu_{p/k})$  is finite over  $k$  of constant rank equal to  $1 + p^f$ .

Let us now assume that  $k$  is an algebraically closed field of characteristic zero, so that  $A[\mathfrak{p}] \simeq (\mathbb{F}_p \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})_k)^2$  is étale. By fixing an arbitrary  $\mathbb{F}_p$ -stable, closed, and  $\lambda$ -isotropic  $k$ -subgroup scheme of  $A[\mathfrak{p}]$  isomorphic to  $\mathbb{F}_p \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})_k$  and considering the corresponding exact sequence, one sees via arguments similar to the ones above that there is an isomorphism  $T \simeq \mathrm{Spec} k \coprod (\mathbb{F}_p \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})_k)$ , so that the fiber  $T$  is still finite over  $k$  of rank  $1 + p^f$ .

We conclude that the morphism  $\pi_1 : \mathcal{M}_c(\mathfrak{p})^{\mathrm{ord}} \rightarrow \mathcal{M}_c^{\mathrm{ord}}$  is quasi-finite, and hence finite since it is proper. Moreover, since the rank of the geometric fibers of  $\pi_1$  is constant and since  $\mathcal{M}_c^{\mathrm{ord}}$  is reduced,  $\pi_1$  is flat by [DR73, V, Lemme 1.13].  $\square$

**Remark 3.8.** When  $g > 1$ , the morphisms  $\pi_1$  and  $\pi_2$  are not finite over the non-ordinary part of  $\mathrm{Sh}$ . This phenomenon already occurs when  $p$  is unramified in  $F$  ([St97]).

For each  $\mathfrak{c} \in \mathfrak{C}$  fix rational, admissible polyhedral cone decompositions for the cusps of the Rapoport locus of  $\mathcal{M}_c(\mathfrak{p})$ . One can then construct toroidal compactifications  $\mathcal{M}_c(\mathfrak{p})^{\mathrm{tor}}, \mathcal{M}(\mathfrak{p})^{\mathrm{tor}}$ , and  $\mathrm{Sh}(\mathfrak{p})^{\mathrm{tor}}$  of the splitting models with Iwahori level structure, as in [RX14<sup>+</sup>, 2.11]. Assuming moreover that these polyhedral cone decompositions are compatible with the ones chosen in 2.3, one obtains morphisms  $\pi_1, \pi_2 : \mathrm{Sh}(\mathfrak{p})^{\mathrm{tor}} \rightarrow \mathrm{Sh}(\mathfrak{p})^{\mathrm{tor}}$  which are finite and flat over the ordinary locus.

**3.9. Construction of  $T_p$ .** We now construct the Hecke operator  $T_p$  over the  $\mathrm{Spec} \mathcal{O}$ -scheme  $\mathrm{Sh}$ , extending a geometric construction of B. Conrad ([Con07, 4.5]).

Recall that for each fractional ideal  $\mathfrak{c} \in \mathfrak{C}$  we fixed an isomorphism  $\theta_c : \mathfrak{c}\mathfrak{p} \rightarrow \mathfrak{c}'$  of fractional ideals with positivity such that  $\mathfrak{c}' \in \mathfrak{C}$ . Moreover we denoted by  $\pi_1 : \mathcal{M}_c(\mathfrak{p}) \rightarrow \mathcal{M}_c$  and  $\pi_2 := \pi_{2, \theta_c} : \mathcal{M}_c(\mathfrak{p}) \rightarrow \mathcal{M}_{c'}$  the “taking the source” and “taking the target” morphisms at the level of fine moduli spaces. Denote by  $f : \mathcal{A}_c \rightarrow \mathcal{M}_c$  the universal abelian scheme over  $\mathcal{M}_c$ , and by  $f_p : \mathcal{A}_c(\mathfrak{p}) \rightarrow \mathcal{M}_c(\mathfrak{p})$  the source of the universal isogeny over  $\mathcal{M}_c(\mathfrak{p})$ . Set  $\dot{\omega} := f_* \Omega_{\mathcal{A}_c/\mathcal{M}_c}^1$  and  $\dot{\omega}_p := f_{p*} \Omega_{\mathcal{A}_c(\mathfrak{p})/\mathcal{M}_c(\mathfrak{p})}^1$ : these are bundles of rank  $g$  over  $\mathcal{M}_c$  and  $\mathcal{M}_c(\mathfrak{p})$  respectively.

Define the following morphism of rank- $g$  bundles over  $\mathcal{M}_c(\mathfrak{p})$ :

$$\alpha' : \pi_2^* \dot{\omega} = \pi_2^* f_* \Omega_{\mathcal{A}_c/\mathcal{M}_c}^1 \rightarrow \dot{\omega}_p \xrightarrow{\sim} \pi_1^* \dot{\omega},$$

where the first arrow is induced by base change and pull-back of differentials, and the second by the contraction isomorphism  $\pi_1^* \Omega_{\mathcal{A}_c/\mathcal{M}_c}^1 \simeq \Omega_{\mathcal{A}_c(\mathfrak{p})/\mathcal{M}_c(\mathfrak{p})}^1$ . There is an analogously defined map at the level of de Rham sheaves:

$$\alpha'' : \pi_2^* \mathcal{H}_{\mathrm{dR}}^1(\mathcal{A}_c/\mathcal{M}_c) \rightarrow \pi_1^* \mathcal{H}_{\mathrm{dR}}^1(\mathcal{A}_c/\mathcal{M}_c),$$

which induces an isomorphism:

$$\pi_2^* \wedge_{\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{M}_c}}^2 \mathcal{H}_{\mathrm{dR}}^1(\mathcal{A}_c/\mathcal{M}_c) \xrightarrow{\sim} \mathfrak{p} \cdot \pi_1^* \wedge_{\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{M}_c}}^2 \mathcal{H}_{\mathrm{dR}}^1(\mathcal{A}_c/\mathcal{M}_c).$$

In particular, the map  $\pi_2^* \dot{\varepsilon}_\tau \rightarrow \pi_1^* \dot{\varepsilon}_\tau$  induced by  $\alpha''$  is an isomorphism if  $\tau \notin \Sigma_p$ , and it is an isomorphism onto  $\tau(\varpi_p) \cdot \pi_1^* \dot{\varepsilon}_\tau$  otherwise.

Fix a paritious weight  $\kappa = ((k_\tau)_{\tau \in \Sigma}, w)$  such that  $k_\tau \geq 1^7$  for all  $\tau \in \Sigma$ . Assume moreover that  $w \geq k_\tau$  if  $\tau \in \Sigma_p$ . Observe that  $\alpha''$  induces an isomorphism

$$\alpha''_\kappa : \bigotimes_{\tau \in \Sigma} \pi_2^* \dot{\varepsilon}_\tau^{\otimes (w - k_\tau)/2} \xrightarrow{\sim} \pi_{p, \kappa} \cdot \bigotimes_{\tau \in \Sigma} \pi_1^* \dot{\varepsilon}_\tau^{\otimes (w - k_\tau)/2}$$

<sup>7</sup>This is necessary to derive the  $q$ -expansion as needed later. Under this condition we have automatically that  $\sum_{\tau \in \Sigma_p} k_\tau \geq ef$ .



where  $\pi_{\mathbf{p},\kappa} := \prod_{\tau \in \Sigma_{\mathbf{p}}} \tau(\omega_{\mathbf{p}})^{(w-k_{\tau})/2}$ . Twisting the sheaf  $\dot{\omega}$  by the character attached to the tuple  $(k_{\tau})_{\tau \in \Sigma}$ , and applying the maps  $\alpha'$  and  $\pi_{\mathbf{p},\kappa}^{-1} \cdot \alpha''_{\kappa}$  one obtains a morphism of invertible sheaves:

$$\alpha : \pi_2^* \dot{\omega}^{\kappa} \rightarrow \pi_1^* \dot{\omega}^{\kappa}.$$

Denote by  $\mathcal{D}$  the relative dualizing sheaf of the smooth scheme  $\mathcal{M}_{\mathbf{c}} \rightarrow \text{Spec } \mathcal{O}$ . The canonical identification  $\mathcal{D} = \bigwedge_{\mathcal{O}_{\mathcal{M}_{\mathbf{c}}}}^g \Omega_{\mathcal{M}_{\mathbf{c}}/\text{Spec } \mathcal{O}}^1$ , together with the existence of the Kodaira-Spencer filtration of  $\Omega_{\mathcal{M}_{\mathbf{c}}/\text{Spec } \mathcal{O}}^1$  ([RX14<sup>+</sup>, Theorem 2.9]), gives rise to a canonical isomorphism

$$KS : \mathcal{D} \xrightarrow{\sim} \dot{\omega}^{(2,0)}.$$

Denote by  $\mathcal{D}_{\mathbf{p}}$  the relative dualizing sheaf of  $\mathcal{M}_{\mathbf{c}}(\mathbf{p}) \rightarrow \text{Spec } \mathcal{O}$ : it exists as an invertible sheaf on  $\mathcal{M}_{\mathbf{c}}(\mathbf{p})$  since the latter is a flat local complete intersection over  $\text{Spec } \mathcal{O}$ , by Proposition 3.7.

We now construct a canonical morphism of sheaves  $\xi : \pi_1^* \mathcal{D} \rightarrow \mathcal{D}_{\mathbf{p}}$  as follows. First, Proposition 3.3 implies that the complement of the  $\mathcal{O}$ -smooth locus  $\mathcal{M}_{\mathbf{c}}(\mathbf{p})^{\text{sm}}$  of  $\mathcal{M}_{\mathbf{c}}(\mathbf{p})$  is of codimension 2 inside  $\mathcal{M}_{\mathbf{c}}(\mathbf{p})$ . Since  $\mathcal{M}_{\mathbf{c}}(\mathbf{p})$  is Cohen-Macaulay, it suffices to construct the desired morphism  $\xi$  over  $\mathcal{M}_{\mathbf{c}}(\mathbf{p})^{\text{sm}}$ . But over  $\mathcal{M}_{\mathbf{c}}(\mathbf{p})^{\text{sm}}$ , the dualizing module  $\mathcal{D}_{\mathbf{p}}$  is given by  $\bigwedge_{\mathcal{O}_{\mathcal{M}_{\mathbf{c}}(\mathbf{p})^{\text{sm}}}}^g \left( \Omega_{\mathcal{M}_{\mathbf{c}}(\mathbf{p})^{\text{sm}}/\text{Spec } \mathcal{O}}^1 \right)$ ; so the natural morphism  $(\pi_1^{\text{sm}})^* \Omega_{\mathcal{M}_{\mathbf{c}}/\text{Spec } \mathcal{O}}^1 \rightarrow \Omega_{\mathcal{M}_{\mathbf{c}}(\mathbf{p})^{\text{sm}}/\text{Spec } \mathcal{O}}^1$  induces the sought-for morphism  $\xi^{\text{sm}}$  between their top exterior powers (here  $\pi_1^{\text{sm}}$  denotes the restriction of  $\pi_1$  to  $\mathcal{M}_{\mathbf{c}}(\mathbf{p})^{\text{sm}}$ ).

We now combine the above maps and denote by  $\eta$  the composition of  $R\pi_{1*} \alpha : R\pi_{1*} \pi_2^* \dot{\omega}^{\kappa} \rightarrow R\pi_{1*} \pi_1^* \dot{\omega}^{\kappa}$  with the following morphism in the derived category  $D_{\text{coh}}^b(\mathcal{M}_{\mathbf{c}})$ :

$$\begin{aligned} R\pi_{1*} \pi_1^* \dot{\omega}^{\kappa} &= \dot{\omega}^{\kappa-(2,0)} \otimes R\pi_{1*} \pi_1^* \dot{\omega}^{(2,0)} \xrightarrow{1 \otimes KS^{-1}} \dot{\omega}^{\kappa-(2,0)} \otimes R\pi_{1*} \pi_1^* \mathcal{D} \\ &\xrightarrow{1 \otimes \xi} \dot{\omega}^{\kappa-(2,0)} \otimes R\pi_{1*} \mathcal{D}_{\mathbf{p}} \xrightarrow{1 \otimes \text{tr}_{\pi_1}} \dot{\omega}^{\kappa-(2,0)} \otimes \mathcal{D} \\ &\xrightarrow{1 \otimes KS} \dot{\omega}^{\kappa-(2,0)} \otimes \dot{\omega}^{(2,0)} = \dot{\omega}^{\kappa}, \end{aligned}$$

where  $\text{tr}_{\pi_1} : R\pi_{1*} \mathcal{D}_{\mathbf{p}} \rightarrow \mathcal{D}$  denotes the trace morphism of normalized dualizing complexes associated to the proper dominant morphism  $\pi_1 : \mathcal{M}_{\mathbf{c}}(\mathbf{p}) \rightarrow \mathcal{M}_{\mathbf{c}}$ . Recall that  $\text{tr}_{\pi_1}$  is non-zero by [BST11<sup>+</sup>, Proposition 2.13], and it is compatible with localizations on the base scheme (*loc.cit.*).

Applying the construction of  $\eta : R\pi_{1*} \pi_2^* \dot{\omega}^{\kappa} \rightarrow \dot{\omega}^{\kappa}$  to each component  $\mathcal{M}_{\mathbf{c}}$  of  $\mathcal{M}$  and quotienting by the action of  $\mathcal{O}_F^{\times,+}/(\mathcal{O}_{F,\mathcal{N}}^{\times})^2$  we obtain a well-defined morphism

$$\eta : R\pi_{1*} \pi_2^* \omega^{\kappa} \rightarrow \omega^{\kappa}$$

in  $D_{\text{coh}}^b(\text{Sh})$ , which extends to a morphism of complexes over  $\text{Sh}^{\text{tor}}$ . Notice that these morphisms over  $\text{Sh}$  and  $\text{Sh}^{\text{tor}}$  do not depend on the choice of identifications  $\theta_{\mathbf{c}} : \mathbf{c}_{\mathbf{p}} \simeq \mathbf{c}'$  for  $\mathbf{c} \in \mathfrak{C}$ .

Now the key technical result is the following

**Proposition 3.10.** *Assume that the weight  $\kappa$  satisfies  $\sum_{\tau \in \Sigma_{\mathbf{p}}} k_{\tau} \geq ef$  and the following conditions:*

$$(3.10.1) \quad \begin{aligned} k_{\tau_{\mathbf{p},j}^{l+1}} &\geq k_{\tau_{\mathbf{p},j}^l} \quad \text{for all } j = 1, \dots, f, \text{ and } l = 1, \dots, e-1; \quad \text{and} \\ pk_{\tau_{\mathbf{p},j}^1} &\geq k_{\tau_{\mathbf{p},j-1}^e} \quad \text{for all } j = 1, \dots, f. \end{aligned}$$

*Then the morphism  $\eta$  in  $D_{\text{coh}}^b(\text{Sh})$  uniquely factors as*

$$(3.10.2) \quad R\pi_{1*} \pi_2^* \omega^{\kappa} \xrightarrow{\frac{1}{p^f} \eta} \omega^{\kappa} \xrightarrow{p^f} \omega^{\kappa},$$

*where the second map is the multiplication by  $p^f$ .*

Note that the condition (3.10.1) automatically forces all  $k_\tau \geq 0$  for  $\tau \in \Sigma_{\mathfrak{p}}$ . Also, we must point out that the condition (3.10.1) is very close to the conjectural ampleness condition in [TX13<sup>+</sup>, Theorem 1.9], at least when  $\mathfrak{p}$  is unramified over  $p$ .

The proof of Proposition 3.10 will be given later in 3.15.<sup>8</sup> We assume it temporarily to construct the operator  $T_{\mathfrak{p}}$  on cohomology. The first map in (3.10.2) gives a canonically defined morphism  $\frac{1}{p^f} \eta : R\pi_{1*} \pi_2^* \omega^\kappa \rightarrow \omega^\kappa$ . For  $m \in \mathbb{Z}_{>0} \cup \{\infty\}$  we denote by  $\omega_m^\kappa$  the sheaf  $\omega^\kappa / (\varpi^m)$ , with the convention that  $\varpi^\infty := 0$ . The morphism  $\frac{1}{p^f} \eta$  induces a morphism

$$\tilde{\eta}_m : R\pi_{1*} \pi_2^* (\omega_m^\kappa) \rightarrow \omega_m^\kappa.$$

**Definition 3.11.** We define the action of the *Hecke operator*  $T_{\mathfrak{p}}$  on the cohomology of  $\omega_m^\kappa$  as the following composition:

$$T_{\mathfrak{p}} : H^i(\mathrm{Sh}, \omega_m^\kappa) \xrightarrow{\pi_2^*} H^i(\mathrm{Sh}(\mathfrak{p}), \pi_2^* \omega_m^\kappa) \xrightarrow{\pi_{1*}} H^i(\mathrm{Sh}, R\pi_{1*} \pi_2^* (\omega_m^\kappa)) \xrightarrow{\tilde{\eta}_m} H^i(\mathrm{Sh}, \omega_m^\kappa).$$

**Remark 3.12.** Let  $R$  be either the ring of integers of a finite extension of  $\mathbb{Q}_l$  for  $(l, p\mathcal{N}) = 1$ , or a field extension of  $\mathbb{Q}_p$ . The moduli schemes  $\mathcal{M}_R$  and  $\mathcal{M}(\mathfrak{p})_R$  defined in the obvious way over  $\mathrm{Spec} R$  are both smooth, and the natural maps  $\pi_{1,R}, \pi_{2,R} : \mathrm{Sh}(\mathfrak{p})_R \rightarrow \mathrm{Sh}_R$  are finite and flat. In particular, one can define the “usual” Hecke operator  $T_{\mathfrak{p},R}$  acting on the cohomology of  $\mathrm{Sh}_R$  by means of the finite-flat trace map attached to  $\pi_{1,R}$ . The compatibility between the finite-flat trace map and the dualizing trace map implies that the operator  $T_{\mathfrak{p}}$  defined above coincides with the classical Hecke operator  $T_{\mathfrak{p},R}$  in these settings.

**Remark 3.13.** The  $q$ -expansion of  $T_{\mathfrak{p}} f$  for  $f \in H^0(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^\kappa(-D))$  is the “expected”  $q$ -expansion. For example, assume that  $p$  is inert in  $F$ ; fix a fractional ideal  $\mathfrak{c} \in \mathfrak{C}$  and an unramified cusp of  $\mathcal{M}_{\mathfrak{c}}$  attached to fractional ideals  $\mathfrak{a}, \mathfrak{b}$  such that  $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$  (cf. [AG05, 6] and [Ka78, 1.1]); we assume that  $\mathfrak{a}$  and  $\mathfrak{b}$  are coprime with  $p$ . For such a cusp, there is a Tate object with additional structure  $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}, \lambda_{\mathrm{can}}, i_{\mathrm{can}}, \eta_{\mathrm{can}}, \xi_{\mathrm{can}})$  defined over a suitable subring of the ring of formal power series  $\mathcal{O}[[q^\alpha : \alpha \in \mathfrak{a}\mathfrak{b}]]$  (this involves a choice of cone decomposition that we omit here since it will not play any role). Evaluating  $f$  at  $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}$  we obtain the  $q$ -expansion of  $f$  at this cusp:  $f(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}, \lambda_{\mathrm{can}}, i_{\mathrm{can}}, \eta_{\mathrm{can}}, \xi_{\mathrm{can}}) = \sum_{\alpha \in (\mathfrak{a}\mathfrak{b})^+} a_\alpha(f, \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}) q^\alpha \in \mathcal{O}/(\varpi^m)[[q^\alpha : \alpha \in (\mathfrak{a}\mathfrak{b})^+]]$ . Suppose that  $f$  has nebentypus  $\varepsilon$ ; then for  $\alpha \in (\mathfrak{a}\mathfrak{b})^+$  we have:

$$a_\alpha(T_{\mathfrak{p}} f, \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}) = a_{p\alpha}(f, \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}) + \varepsilon(p) \prod_{\tau \in \Sigma} p^{k_\tau - 1} a_{p^{-1}\alpha}(f, \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}).$$

Notice that the formula is meaningful in  $\mathcal{O}/(\varpi^m)$  since  $k_\tau \geq 1$ . Moreover, recall that we have introduced a normalization by the factor  $\pi_{\mathfrak{p},\kappa}$  in the definition of our Hecke operators.

**Remark 3.14.** The morphism  $KS \circ (\mathrm{tr}_{\pi_1} \circ \xi) \circ KS^{-1} : R\pi_{1*} \pi_1^* \dot{\omega}^{(2,0)} \rightarrow \dot{\omega}^{(2,0)}$  that appears in the composition defining  $\eta$  coincides, over an open subscheme of  $\mathcal{M}_{\mathfrak{c}}$  on which the map  $\pi_1$  is finite flat, with the usual finite flat trace map  $\pi_{1*} \pi_1^* \dot{\omega}^{(2,0)} \rightarrow \dot{\omega}^{(2,0)}$ . This follows from the compatibility between the dualizing trace map and the finite flat trace map (cf. [Con00]). In particular, when  $g = 1$  (so that  $\pi_1$  is finite flat over the entire space  $\mathcal{M} = \mathcal{M}_{\mathfrak{c}}$ ), our construction *coincides* with the construction given in [Con07, 4.5].

**3.15. Outline of the proof of Proposition 3.10.** Since the morphism  $\eta$  is obtained by taking invariants under  $\mathcal{O}_F^{\times,+} / (\mathcal{O}_{F,\mathcal{N}}^{\times})^2$  of the (homonymous) morphism  $R\pi_{1*} \pi_2^* \dot{\omega}^\kappa \rightarrow \dot{\omega}^\kappa$  on  $\mathcal{M}$ , it suffices to prove the result for the latter morphism, as a morphism in  $D_{\mathrm{coh}}^b(\mathcal{M})$ . This will follow from the three propositions below.

<sup>8</sup>After correcting the gap pointed out by Pilloni, we were informed by him that he has a different and simpler proof of Proposition 3.10. But our proof reveals some interesting finer geometry of the map  $\pi_1 : \mathcal{M}(\mathfrak{p}) \rightarrow \mathcal{M}$ .

**Proposition 3.16.** *Suppose that  $\sum_{\tau \in \Sigma_{\mathbb{p}}} k_{\tau} \geq ef$ . Restricting  $\eta$  to  $\mathcal{M}^{\text{ord}}$ , the homomorphism  $\eta : (\pi_{1*}\pi_2^*\dot{\omega}^{\kappa})|_{\mathcal{M}^{\text{ord}}} \rightarrow \dot{\omega}^{\kappa}|_{\mathcal{M}^{\text{ord}}}$  of coherent sheaves<sup>9</sup> factors uniquely as*

$$(\pi_{1*}\pi_2^*\dot{\omega}^{\kappa})|_{\mathcal{M}^{\text{ord}}} \rightarrow \dot{\omega}^{\kappa}|_{\mathcal{M}^{\text{ord}}} \xrightarrow{\cdot p^f} \dot{\omega}^{\kappa}|_{\mathcal{M}^{\text{ord}}}.$$

**Proposition 3.17.** *Suppose that the support of  $R^t\pi_{1*}\pi_2^*\dot{\omega}_{\mathbb{F}}^{\kappa}$  has codimension at least  $t+1$  in the special fiber  $\mathcal{M}_{\mathbb{F}}$  for all  $t \geq 1$ . Then  $\eta$  factors uniquely as*

$$(3.17.1) \quad R\pi_{1*}\pi_2^*\dot{\omega}^{\kappa} \rightarrow \dot{\omega}^{\kappa} \xrightarrow{\cdot p^f} \dot{\omega}^{\kappa}$$

in  $D_{\text{coh}}^b(\mathcal{M})$  if the restriction of  $\eta$  does on the ordinary locus (as shown in Proposition 3.16).

**Proposition 3.18.** *Suppose that the weight  $\kappa$  satisfies the condition (3.10.1). Then the assumption in Proposition 3.17 holds.*

Proposition 3.17 is a formal homological algebra result. Proposition 3.16 is a straightforward generalization of [Con07, Theorem 4.5.1]. The proof of these two propositions will be given shortly in 3.19 and 3.20, after we summarize the basic idea of the proof of Proposition 3.18 (whose proof will be given in Section 4).

By Grothendieck's formal function theorem, the cohomological dimension of  $R\pi_{1*}\pi_2^*\dot{\omega}_{\mathbb{F}}^{\kappa}$  is bounded by the dimension of the fiber of the map  $\pi_1 : \mathcal{M}(\mathfrak{p})_{\mathbb{F}} \rightarrow \mathcal{M}_{\mathbb{F}}$ . Since the source and the target of  $\pi_1$  have the same dimension, localizing at a codimension  $t$  point  $x$  of  $\mathcal{M}_{\mathbb{F}}$ ,  $R^{>t}\pi_{1*}\pi_2^*\dot{\omega}_{\mathbb{F}}^{\kappa}$  is automatically trivial, and the contribution to the localization of  $R^t\pi_{1*}\pi_2^*\dot{\omega}_{\mathbb{F}}^{\kappa}$  at  $x$  only comes from the big irreducible components of  $\mathcal{M}(\mathfrak{p})_{\mathbb{F}}$ . Recall that  $\mathcal{M}_{\mathbb{F}}$  admits the Goren-Oort stratification (cf. 2.7). The proof of Proposition 3.18 consists of the following ingredients.

- The image of each irreducible component of  $\mathcal{M}(\mathfrak{p})_{\mathbb{F}}$  is a closed Goren-Oort stratum (Proposition 4.5); so it suffices to look at the those open Goren-Oort stratum  $X_{\mathbb{T}}^{\circ}$  whose closure is the image of some irreducible component of  $\mathcal{M}(\mathfrak{p})_{\mathbb{F}}$ ;
- over the *geometric* generic point of the open stratum  $X_{\mathbb{T}}^{\circ}$ , only the dimension  $t = \#\mathbb{T}$  fibers are relevant (by Proposition 4.8), namely the ones given by the irreducible components of  $\mathcal{M}(\mathfrak{p})_{\mathbb{F}}$ ; and
- these dimension  $t$  fibers are unions of products of  $\mathbb{P}^1$ 's, such that the restriction of  $\pi_2^*\dot{\omega}^{\kappa}$  is  $\mathcal{O}(n)$  on each  $\mathbb{P}^1$ -factor for some  $n \geq -1$  (cf. 4.9)

**3.19. Proof of Proposition 3.16.** It is enough to check for the stalks at each closed point of  $\mathcal{M}^{\text{ord}}$ . Let  $k$  be a separably closed extension of  $\mathbb{F}$  and let  $x : \text{Spec } k \rightarrow \mathcal{M}^{\text{ord}}$  be a map of  $\text{Spec } \mathcal{O}$ -schemes. Let  $y : \text{Spec } k \rightarrow \mathcal{M}(\mathfrak{p})^{\text{ord}}$  be an element in the scheme-theoretic fiber  $\pi_1^{-1}(x)$ . Denote by  $R_x$  the strictly henselian local ring of the stalk of  $\mathcal{O}_{\mathcal{M}}$  at  $x$ , and similarly for  $R_y$  and  $R_{\pi_2(y)}$ .

By Proposition 3.7, the morphism  $\pi_1$  is finite flat over the ordinary locus of  $\mathcal{M}_{\mathbb{F}}$ , so that the fiber  $\pi_1^{-1}(x)$  is a finite scheme over  $\text{Spec } k$  and the map  $R_x \rightarrow R_y$  is finite flat. By Remark 3.14 it then suffices to show that the composition:

$$(3.19.1) \quad \dot{\omega}_{\pi_2(y)}^{\kappa} \xrightarrow{\eta_y^{\kappa}} \dot{\omega}_{\mathfrak{p},y}^{\kappa} \simeq (\pi_1^*\dot{\omega}^{\kappa})_y = R_y \otimes_{R_x} \dot{\omega}_x^{\kappa} \xrightarrow{\text{Tr}_{y|x} \otimes 1} \dot{\omega}_x^{\kappa}$$

has image in  $p^f \dot{\omega}_x^{\kappa}$ . Here  $\eta_y^{\kappa}$  is induced by pulling back differentials, the isomorphism is induced by the “contraction map”, and  $\text{Tr}_{y|x} : R_y \rightarrow R_x$  is the  $R_x$ -linear *finite flat* trace map. In particular, we don't need to be worried about the Kodaira-Spencer isomorphism!

Assume that  $x$  corresponds to the abelian scheme  $A/\text{Spec } k$  with extra structure, and that  $y$  corresponds to an isogeny  $A \rightarrow A'$  defined over  $\text{Spec } k$  and with kernel  $C$  (notice we can forget about the filtrations because we are working over the ordinary locus). The closed, finite-flat  $k$ -group scheme  $C \subset A[\mathfrak{p}]$  has rank  $p^f$  and it comes with an action of  $\mathcal{O}_F/\mathfrak{p} =: \mathbb{F}_{\mathfrak{p}}$ .

<sup>9</sup>Note that  $\pi_1$  is finite and flat on  $\mathcal{M}^{\text{ord}}$  by the proof of Proposition 3.7; so there is no higher derived pushforward.

We now prove the desired result distinguishing two cases, depending on whether  $C$  is étale or multiplicative (no other possibilities occur in our settings: remember that  $A$  is ordinary). We work with the strict henselization of the stalks modulo  $p^f$ .

Case 1:  $C$  is multiplicative. In this case the kernel of the universal isogeny  $\phi$  over  $R_y/(p^f)$  is isomorphic to  $\mathbb{F}_p \otimes_{\mathbb{Z}} \mu_{p/R_y/(p^f)}$ . Recall that  $\Sigma_p$  denotes the set of field embeddings  $F \rightarrow \overline{\mathbb{Q}_p}$  inducing the  $p$ -adic place  $p$ . For each  $\tau \in \Sigma_p$  the pull-back map  $(\dot{\omega}_{\pi_2(y)})_\tau \rightarrow (\dot{\omega}_{p,y})_\tau$  is zero modulo  $\tau(\varpi_p)$ . In particular,  $\eta_y^\kappa$  is zero modulo  $p^f$  as desired, since hence  $\sum_{\tau \in \Sigma_p} k_\tau \geq ef$ .

Case 2:  $C$  is étale. The  $\hat{R}_x/(p^f)$ -algebra  $\hat{R}_y/(p^f)$  (the “hat” denotes completion) classifies splittings of the connected-étale sequence of  $\mathcal{A}[p]$ , where  $\mathcal{A}$  is the universal ordinary abelian scheme over  $\hat{R}_x/(p^f)$ . These splittings are a torsor under the group-scheme

$$\mathrm{Hom}_{\mathbb{F}_p, \mathrm{Spec}(\hat{R}_x/(p^f))}(\mathcal{A}[p]^{\mathrm{ét}}, \mathcal{A}[p]^\circ) \simeq \mathbb{F}_p \otimes_{\mathbb{Z}} \mu_{p/\hat{R}_x/(p^f)}$$

defined over  $\hat{R}_x/(p^f)$ . Such torsors are classified by  $H_{\mathrm{fppf}}^1(\mathrm{Spec} \hat{R}_x/(p^f), \mathbb{F}_p \otimes_{\mathbb{Z}} \mu_{p/\hat{R}_x/(p^f)})$ , which by Kummer theory is equal to  $\mathbb{F}_p \otimes_{\mathbb{Z}} (\hat{R}_x/(p^f))^\times / (\hat{R}_x/(p^f))^{\times p}$ . Therefore there are units  $u_i \in (\hat{R}_x/(p^f))^\times$  which are not  $p$ th powers such that

$$\hat{R}_y/(p^f) \simeq \frac{\hat{R}_x/(p^f)[X_1, \dots, X_f]}{(X_1^p - u_1, \dots, X_f^p - u_f)}.$$

This implies that the  $\hat{R}_x/(p^f)$ -linear trace map  $\hat{R}_y/(p^f) \rightarrow \hat{R}_x/(p^f)$  is zero: the only thing to check is the vanishing of the trace on the identity element, which occurs since  $\hat{R}_y/(p^f)$  has rank  $p^f$  as an  $\hat{R}_x/(p^f)$ -module. We deduce that also the trace map  $\mathrm{Tr}_{y|x} : R_y/(p^f) \rightarrow R_x/(p^f)$  is zero, and hence the map (3.19.1) vanishes modulo  $p^f$ , as desired.  $\square$

**3.20. Proof of Proposition 3.17.** Using the exact sequence

$$0 \rightarrow \pi_2^* \dot{\omega}^\kappa \xrightarrow{\cdot \varpi} \pi_2^* \dot{\omega}^\kappa \longrightarrow \pi_2^* \dot{\omega}_{\mathbb{F}}^\kappa \rightarrow 0,$$

we see that, if  $R^t \pi_{1,*} \pi_2^* \dot{\omega}_{\mathbb{F}}^\kappa$  (for  $t > 0$ ) is zero when localizing at a codimension  $t$  point  $x \in \mathcal{M}_{\mathbb{F}}$ , then multiplication by  $\varpi$  is surjective on  $R^t \pi_{1,*} \pi_2^* \dot{\omega}^\kappa$  when localized at  $x$ . By Nakayama’s Lemma, this means that  $R^t \pi_{1,*} \pi_2^* \dot{\omega}^\kappa$  is trivial when localized at  $x$ , as  $R^t \pi_{1,*} \pi_2^* \dot{\omega}^\kappa$  is a coherent sheaf. So the condition of this Proposition implies that

- the (set-theoretical) support of  $R^t \pi_{1,*} \pi_2^* \dot{\omega}^\kappa$  has codimension  $> t$  in the special fiber  $\mathcal{M}_{\mathbb{F}}$  for all  $t \geq 1$ .

For two complexes  $C^\bullet, D^\bullet \in D^b(\mathcal{M})$ , we may define the sheaf of morphisms  $\mathcal{H}om_{D^b(\mathcal{M})}(C^\bullet, D^\bullet)$  as the (coherent) sheaf on  $\mathcal{M}$  whose evaluation on open affine subspaces  $\mathcal{U} \subseteq \mathcal{M}$  is

$$\mathrm{Hom}_{D^b(\mathcal{U})}(C^\bullet \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{U}}, D^\bullet \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{U}}).$$

For  $i \in \mathbb{Z}$ , we write  $\mathcal{H}om_{D^b(\mathcal{M})}^i(C^\bullet, D^\bullet)$  for  $\mathcal{H}om_{D^b(\mathcal{M})}(C^\bullet, D^\bullet[i])$ .

Consider the long exact sequence obtained by applying the functor  $\mathcal{H}om_{D^b(\mathcal{M})}^\bullet(R\pi_{1,*} \pi_2^* \dot{\omega}^\kappa, -)$  to  $0 \rightarrow \dot{\omega}^\kappa \xrightarrow{\cdot p^f} \dot{\omega}^\kappa \rightarrow \dot{\omega}^\kappa/p^f \rightarrow 0$ . To show that the map  $\eta$  factors uniquely as (3.17.1), it is enough to show that

- (existence of factorization) the composition  $R\pi_{1,*} \pi_2^* \dot{\omega}^\kappa \xrightarrow{\eta} \dot{\omega}^\kappa \rightarrow \dot{\omega}^\kappa/p^f$  is the zero element in  $\mathcal{H}om_{D^b(\mathcal{M})}(R\pi_{1,*} \pi_2^* \dot{\omega}^\kappa, \dot{\omega}^\kappa/p^f)$ , and
- (uniqueness of the factorization)  $\mathcal{H}om_{D^b(\mathcal{M})}^{-1}(R\pi_{1,*} \pi_2^* \dot{\omega}^\kappa, \dot{\omega}^\kappa/p^f) = 0$ .

We claim that the natural map  $\pi_{1,*} \pi_2^* \dot{\omega}^\kappa \rightarrow R\pi_{1,*} \pi_2^* \dot{\omega}^\kappa$  induces an injection

$$\mathcal{H}om_{D^b(\mathcal{M})}^i(R\pi_{1,*} \pi_2^* \dot{\omega}^\kappa, \dot{\omega}^\kappa/p^f) \hookrightarrow \mathcal{H}om_{D^b(\mathcal{M})}^i(\pi_{1,*} \pi_2^* \dot{\omega}^\kappa, \dot{\omega}^\kappa/p^f) \cong \mathcal{H}om_{\mathcal{O}_{\mathcal{M}}}^i(\pi_{1,*} \pi_2^* \dot{\omega}^\kappa, \dot{\omega}^\kappa/p^f),$$

for  $i = -1, 0$ . Here  $\mathcal{H}om_{\mathcal{O}_{\mathcal{M}}}^{-1}(\pi_{1,*}\pi_2^*\dot{\omega}^\kappa, \dot{\omega}^\kappa/p^f)$  is automatically zero because  $\mathcal{H}om$  is left exact; so (a) follows from this injectivity. Similarly, when  $i = 0$ , the injectivity implies that a map in  $\mathcal{H}om_{D^b(\mathcal{M})}(R\pi_{1,*}\pi_2^*\dot{\omega}^\kappa, \dot{\omega}^\kappa/p^f)$  is zero if and only if its induced map in  $\mathcal{H}om_{\mathcal{O}_{\mathcal{M}}}(\pi_{1,*}\pi_2^*\dot{\omega}^\kappa, \dot{\omega}^\kappa/p^f)$  is zero, which is in turn equivalent to its restriction to  $\mathcal{M}^{\text{ord}}$  is zero. So this together with the assumption of the Proposition implies (b). In summary, it is enough to prove the claim on injectivity.

Write  $C^\bullet$  for  $R\pi_{1,*}\pi_2^*\dot{\omega}^\kappa$ . For  $t \in \mathbb{Z}_{\geq 0}$ , we denote by  $\tau_{\leq t}C^\bullet$  its truncation at degree  $\leq t$ , that is, the unique element in  $D^b(\mathcal{M})$  (up to quasi-isomorphism) such that  $H^{>t}(\tau_{\leq t}C^\bullet) = 0$ , and there is a map  $\tau_{\leq t}C^\bullet \rightarrow C^\bullet$  inducing isomorphisms on the cohomology groups with degree  $\leq t$ . In particular,  $\tau_{\leq 0}C^\bullet \cong \pi_{1,*}\pi_2^*\dot{\omega}^\kappa$  and  $\tau_{\leq t}C^\bullet \cong C^\bullet$  for  $t \gg 0$ .

To (inductively) prove the claim, it suffices to prove that the following map is injective for every  $t \geq 1$  and  $i = -1, 0$ :

$$\mathcal{H}om_{D^b(\mathcal{M})}^i(\tau_{\leq t}C^\bullet, \dot{\omega}^\kappa/p^f) \hookrightarrow \mathcal{H}om_{D^b(\mathcal{M})}^i(\tau_{\leq t-1}C^\bullet, \dot{\omega}^\kappa/p^f).$$

Using the long exact sequence associated to the tautological exact triangle

$$\tau_{\leq t-1}C^\bullet \rightarrow \tau_{\leq t}C^\bullet \rightarrow R^t\pi_{1,*}\pi_2^*\dot{\omega}^\kappa[-t] \xrightarrow{+1},$$

it suffices to show that

$$\mathcal{E}xt_{\mathcal{O}_{\mathcal{M}}}^j(R^t\pi_{1,*}\pi_2^*\dot{\omega}^\kappa, \dot{\omega}^\kappa/p^f) = 0 \quad \text{for } j = t-1, t.$$

Looking at the long exact sequence induced by the following exact sequence  $0 \rightarrow \dot{\omega}^\kappa \xrightarrow{\cdot p^f} \dot{\omega}^\kappa \rightarrow \dot{\omega}^\kappa/p^f \rightarrow 0$ , it is enough to show that

$$\mathcal{E}xt_{\mathcal{O}_{\mathcal{M}}}^i(R^t\pi_{1,*}\pi_2^*\dot{\omega}^\kappa, \dot{\omega}^\kappa) = 0. \quad \text{for } i = t-1, t, t+1.$$

Since  $\mathcal{M}$  is regular, the condition that the (set theoretical) support of  $R^t\pi_{1,*}\pi_2^*\dot{\omega}^\kappa$  has codimension  $> t$  in  $\mathcal{M}_{\mathbb{F}}$  (and hence codimension  $> t+1$  in  $\mathcal{M}$ ), implies that

$$\mathcal{E}xt_{\mathcal{O}_{\mathcal{M}}}^{\leq t+1}(R^t\pi_{1,*}\pi_2^*\dot{\omega}^\kappa, \dot{\omega}^\kappa) = 0.$$

This completes the proof of Proposition 3.17.  $\square$

#### 4. PROOF OF PROPOSITION 3.18

This section is entirely devoted to proving Proposition 3.18. Readers may choose to skip this technical part to the last section directly. We keep the notation from the previous sections.

**4.1. Recollection of the moduli problems for  $\mathcal{M}$  and  $\mathcal{M}(\mathfrak{p})$ .** We first recall from the definition of  $\mathcal{M}$  in 2.2 that, for the universal abelian variety  $\mathcal{A}$  over  $\mathcal{M}$ , we have a universal filtration

$$0 = \mathcal{F}_{\mathfrak{p}_{i,j}}^{(0)} \subsetneq \mathcal{F}_{\mathfrak{p}_{i,j}}^{(1)} \subsetneq \cdots \subsetneq \mathcal{F}_{\mathfrak{p}_{i,j}}^{(e_i)} = \omega_{\mathcal{A}/\mathcal{M}, \mathfrak{p}_{i,j}} \quad \text{for } i = 1, \dots, r; \quad j = 1, \dots, f_i,$$

with subquotients  $\dot{\omega}_{\tau_{\mathfrak{p}_{i,j}}^l}$ . For  $l = 1, \dots, e_i$ , we put

$$\mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)} := \{z \in (\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)})^\perp / \mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)} \mid [\varpi_i]z - \tau_{\mathfrak{p}_{i,j}}^l(\varpi_i)z = 0\},$$

where  $(\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)})^\perp$  is the orthogonal complement of  $\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)}$  in  $H_{\text{dR}}^1(\mathcal{A}/\mathcal{M})_{\mathfrak{p}_{i,j}}$  with respect to the natural pairing induced by the polarization. By [RX14<sup>+</sup>, Corollary 2.10], we have

- (1) each  $\mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)}$  is a locally free coherent sheaf of rank two over  $\mathcal{M}$ ,
- (2)  $\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)} / \mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)}$  is a rank one subbundle of  $\mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)}$ ,<sup>10</sup> and
- (3) there is a canonical isomorphism  $\wedge_{\mathcal{O}_{\mathcal{M}}}^2 \mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)} \cong \dot{\epsilon}_{\tau_{\mathfrak{p}_{i,j}}^l}$ .

<sup>10</sup>This is in fact a corollary of the previous point because  $\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)} / \mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)}$  is a subbundle of  $H_{\text{dR}}^1(\mathcal{A}/\mathcal{M})_{\mathfrak{p}_{i,j}}$  contained in  $\mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)}$  (since  $[\varpi_i] - \tau_{\mathfrak{p}_{i,j}}^{(l)}(\varpi_i)$  kills  $\mathcal{F}_{\mathfrak{p}_{i,j}}^{(l)} / \mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)}$ ).

Moreover, by [RX14<sup>+</sup>, Constructions 3.3 and 3.6], the partial Hasse invariants we recalled in 2.6 extend to *surjective* homomorphisms over  $\mathcal{M}_{\mathbb{F}}$

$$(4.1.1) \quad m_{\varpi_{i,j}}^{(l)} : \mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)} \twoheadrightarrow \mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-1)} / \mathcal{F}_{\mathfrak{p}_{i,j}}^{(l-2)} \subset \mathcal{H}_{\mathfrak{p}_{i,j}}^{(l-1)} \quad \text{and} \quad \text{Hasse}_{\varpi_{i,j}} : \mathcal{H}_{\mathfrak{p}_{i,j}}^{(1)} \twoheadrightarrow \dot{\omega}_{\tau_{\mathfrak{p}_{i,j}-1}}^{\otimes p} \subset (\mathcal{H}_{\mathfrak{p}_{i,j}}^{(e_i)})^{(p)},$$

for  $i = 1, \dots, r$ ,  $j = 1, \dots, f_i$ , and  $l = 2, \dots, e_i$ .

Now, we recall from the definition of  $\mathcal{M}(\mathfrak{p})$  in 3.1 that, over  $\mathcal{M}(\mathfrak{p})$ , we have the following two isogenies of universal abelian varieties.

$$\phi : \mathcal{A} \rightarrow \mathcal{A}' \quad \text{and} \quad \psi : \mathcal{A}' \rightarrow \mathcal{A}.$$

We write  $\mathcal{F}_{\mathfrak{p}_{i,j}}'^{(l)}$  and  $\mathcal{H}_{\mathfrak{p}_{i,j}}'^{(l)}$  for the corresponding constructions for  $\mathcal{A}'$ . By the proof of Proposition 3.3, we see that  $\phi^*$  and  $\psi^*$  induce homomorphisms

$$(4.1.2) \quad \phi_{\mathfrak{p}_{i,j},l}^* : \mathcal{H}_{\mathfrak{p}_{i,j}}'^{(l)} \rightarrow \mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)} \quad \text{and} \quad \psi_{\mathfrak{p}_{i,j},l}^* : \mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)} \rightarrow \mathcal{H}_{\mathfrak{p}_{i,j}}'^{(l)},$$

such that  $\phi_{\mathfrak{p}_{i,j},l}^*$  and  $\psi_{\mathfrak{p}_{i,j},l}^*$  are isomorphisms if  $\mathfrak{p}_i \neq \mathfrak{p}$ , and, over  $\mathcal{M}_{\mathbb{F}}$ , we have

$$\text{Im}(\phi_{\mathfrak{p},j,l}^*) = \text{Ker}(\psi_{\mathfrak{p},j,l}^*) \quad \text{and} \quad \text{Im}(\psi_{\mathfrak{p},j,l}^*) = \text{Ker}(\phi_{\mathfrak{p},j,l}^*)$$

and both modules are subbundles of rank one of the corresponding rank two vector bundles over  $\mathcal{M}_{\mathbb{F}}$ . One can check from the definition that  $\phi_{\mathfrak{p}_{i,j},l}^*$  and  $\psi_{\mathfrak{p}_{i,j},l}^*$  respect the partial Hasse invariant maps (4.1.1), namely we have commutative diagrams of sheaves over  $\mathcal{M}(\mathfrak{p})_{\mathbb{F}}$

$$\begin{array}{ccccc} \mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)} & \xrightarrow{\psi_{\mathfrak{p}_{i,j},l}^*} & \mathcal{H}_{\mathfrak{p}_{i,j}}'^{(l)} & \xrightarrow{\phi_{\mathfrak{p}_{i,j},l}^*} & \mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)} \\ \downarrow m_{\varpi_{i,j}}^{(l)} & & \downarrow m_{\varpi_{i,j}}^{(l)} & & \downarrow m_{\varpi_{i,j}}^{(l)} \\ \mathcal{H}_{\mathfrak{p}_{i,j}}^{(l-1)} & \xrightarrow{\psi_{\mathfrak{p}_{i,j},l}^*} & \mathcal{H}_{\mathfrak{p}_{i,j}}'^{(l-1)} & \xrightarrow{\phi_{\mathfrak{p}_{i,j},l}^*} & \mathcal{H}_{\mathfrak{p}_{i,j}}^{(l-1)} \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mathcal{H}_{\mathfrak{p}_{i,j}}^{(1)} & \xrightarrow{\psi_{\mathfrak{p}_{i,j},l}^*} & \mathcal{H}_{\mathfrak{p}_{i,j}}'^{(1)} & \xrightarrow{\phi_{\mathfrak{p}_{i,j},l}^*} & \mathcal{H}_{\mathfrak{p}_{i,j}}^{(1)} \\ \downarrow \text{Hasse}_{\varpi_{i,j}} & & \downarrow \text{Hasse}_{\varpi_{i,j}} & & \downarrow \text{Hasse}_{\varpi_{i,j}} \\ (\mathcal{H}_{\mathfrak{p}_{i,j}}^{(e_i)})^{(p)} & \xrightarrow{\psi_{\mathfrak{p}_{i,j},l}^*} & (\mathcal{H}_{\mathfrak{p}_{i,j}}'^{(e_i)})^{(p)} & \xrightarrow{\phi_{\mathfrak{p}_{i,j},l}^*} & (\mathcal{H}_{\mathfrak{p}_{i,j}}^{(e_i)})^{(p)}, \end{array}$$

for  $i = 1, \dots, r$ ,  $j = 1, \dots, f_i$ , and  $l = 2, \dots, e_i$ .

**4.2. Frobenius factor.** Let  $k$  be a perfect field. A map  $g : Y \rightarrow X$  of  $k$ -schemes is called a *Frobenius factor* if it induces bijection on closed points, and there are an integer  $s \in \mathbb{N}$  and a morphism  $g' : X^{(p^s)} \rightarrow Y$  such that the composition  $g \circ g'$  is the relative  $p^s$ -Frobenius on  $X$ , where  $X^{(p^s)} := X \times_{k, \text{Frob}_{p^s}} k$ .

It is proved in [He12, Proposition 4.8] that if  $g : Y \rightarrow X$  is a proper morphism of  $k$ -schemes of finite type that induces bijection on geometric points and if  $Y$  is reduced and  $X$  is normal, then  $g$  is a Frobenius factor.

**Notation 4.3.** For a subset  $\mathbf{S} \subseteq \Sigma_{\mathfrak{p}}$ , we write  $\mathbf{S}^c$  for  $\Sigma_{\mathfrak{p}} \setminus \mathbf{S}$ . Recall that  $e$  and  $f$  denote the ramification and inertia degree of  $\mathfrak{p}$  over  $p$ .

Recall that, elements in  $\Sigma_{\mathfrak{p}}$  come with a chosen order, as fixed in 2.1. We consider the map  $\theta : \Sigma_{\mathfrak{p}} \rightarrow \Sigma_{\mathfrak{p}}$  given by  $\theta(\tau_{\mathfrak{p},j}^l) = \tau_{\mathfrak{p},j}^{l+1}$  if  $l < e$  and  $\theta(\tau_{\mathfrak{p},j}^e) = \tau_{\mathfrak{p},j+1}^1$ . When  $\mathfrak{p}$  is unramified over  $p$ ,  $\theta$  is exactly the action of Frobenius  $\sigma$  on  $\Sigma_{\mathfrak{p}}$ .<sup>11</sup> This way, the partial Hasse invariant map at  $\tau$  (4.1.1) is a map

$$\dot{\omega}_{\tau} \longrightarrow \dot{\omega}_{\theta^{-1}\tau} \quad \text{or} \quad \dot{\omega}_{\theta^{-1}\tau}^{\otimes p}.$$

To better imitate the unramified case, we rename  $\tau_{\mathfrak{p},j}^l$  into  $\tau_{(j-1)e+l}$  and set  $\tau_a = \tau_{a \bmod ef}$ , so that we have  $\theta(\tau_a) = \tau_{a+1}$ . Accordingly, we write  $\mathcal{H}_{\tau_{(j-1)e+l}}^{(l)}$  for  $\mathcal{H}_{\mathfrak{p}_{i,j}}^{(l)}$ , write  $\phi_{\tau_{(j-1)e+l}}^*$  for  $\phi_{\mathfrak{p}_{i,j},l}^*$ , write  $\psi_{\tau_{(j-1)e+l}}^*$  for  $\psi_{\mathfrak{p}_{i,j},l}^*$ , and write  $\text{Ha}_{\tau_{(j-1)e+l}}$  for the corresponding partial Hasse invariant map

<sup>11</sup>We recommend the readers to assume that  $\mathfrak{p}$  is unramified over  $p$  when reading this section first time. It helps to understand the key point of the argument.

(4.1.1) on  $\mathcal{A}_{\mathbb{F}}$ , namely  $m_{\varpi,j}^{(l)}$  if  $l \neq 1$  and  $\text{Hasse}_{\varpi,j}$  if  $l = 1$ . Similarly,  $\text{Ha}'_{\tau}$  denotes the partial Hasse invariant maps between  $\mathcal{H}'_{\tau}$ 's that are defined for  $\mathcal{A}'_{\mathbb{F}}$ .

**4.4. A combinatorics construction.** For later convenience, we make explicit the following combinatorics construction: for each  $\mathbf{S}$  a nonempty proper subset of  $\Sigma_{\mathbf{p}}$ , we decompose the set  $\mathbf{S}$  into the union of “adjacent places” (according to the action of  $\theta$  on  $\Sigma_{\mathbf{p}}$ ), namely, we write

$$(4.4.1) \quad \mathbf{S} = \{\tau_{a_t-1}, \dots, \tau_{a_t-\lambda_t}, \tau_{a_{t-1}-1}, \dots, \tau_{a_{t-1}-\lambda_{t-1}}, \dots, \tau_{a_1-1}, \dots, \tau_{a_1-\lambda_1}\},$$

where the numbers

- (1)  $a_1, \dots, a_t \in \{1, \dots, ef\}$  are in increasing order such that  $a_{i+1} - a_i \geq 2$  for all  $i \geq 2$  and  $a_1 + ef - a_t \geq 2$ , and
- (2)  $\lambda_i \in \{1, \dots, a_i - a_{i-1} - 1\}$  for all  $i \geq 2$  and  $\lambda_1 \in \{1, \dots, a_1 + ef - a_t - 1\}$ .

Then we have

$$\mathbf{T} := \theta(\mathbf{S}) \setminus \mathbf{S} = \{\tau_{a_1}, \dots, \tau_{a_t}\}, \quad \text{and} \quad \mathbf{S} \setminus \theta(\mathbf{S}) = \{\tau_{a_1-\lambda_1}, \tau_{a_2-\lambda_2}, \dots, \tau_{a_t-\lambda_t}\}.$$

**Proposition 4.5.** *Let  $\mathbf{S}$  be a subset of  $\Sigma_{\mathbf{p}}$ , and put  $\mathbf{T} := \theta(\mathbf{S}) \setminus \mathbf{S}$ . Then  $\pi_1(Y_{\mathbf{S}, \mathbf{S}^c}) = X_{\mathbf{T}}$ . More precisely, if we write  $\text{pr} : Y'_{\mathbf{S}, \mathbf{S}^c} \rightarrow X_{\mathbf{T}}$  for the  $(\mathbb{P}^1)^{\# \mathbf{T}}$ -bundle given by  $\prod_{\tau \in \mathbf{S} \setminus \theta(\mathbf{S})} \mathbb{P}(\mathcal{H}_{\tau})$  (where the product is taken over  $X_{\mathbf{T}}$ ), then  $\pi_1|_{Y_{\mathbf{S}, \mathbf{S}^c}}$  factors as*

$$Y_{\mathbf{S}, \mathbf{S}^c} \xrightarrow{g} Y'_{\mathbf{S}, \mathbf{S}^c} \xrightarrow{\text{pr}} X_{\mathbf{T}},$$

where  $g$  is a Frobenius factor.

*Proof.* The cases of  $\mathbf{S} = \emptyset$  and  $\mathbf{S} = \Sigma_{\mathbf{p}}$  correspond to the closure of the ordinary locus of  $\mathcal{M}(\mathbf{p})_{\mathbb{F}}$ . In this case,  $\pi_1$  is the Frobenius map and an isomorphism, respectively. From now on, we assume that  $\mathbf{S}$  is a nonempty proper subset of  $\Sigma_{\mathbf{p}}$ . We may assume that  $\mathbf{S}$  is as described in Subsection 4.4 so that  $\mathbf{T} = \{\tau_{a_1}, \dots, \tau_{a_t}\}$ .

We first show that  $\pi_1(Y_{\mathbf{S}, \mathbf{S}^c}) \subseteq X_{\mathbf{T}}$ . Take an  $S$ -point of  $Y_{\mathbf{S}, \mathbf{S}^c}$ , and we write  $\mathcal{H}_{\tau}$  and  $\mathcal{H}'_{\tau}$  for the evaluation of  $\mathcal{H}_{\tau}$  and  $\mathcal{H}'_{\tau}$  at this  $S$ -point. For  $\tau \in \mathbf{T} = \mathbf{S} \setminus \theta(\mathbf{S})$ , namely  $\tau \in \mathbf{S}$  and  $\theta^{-1}\tau \in \mathbf{S}^c$ , consider the following commutative diagram

$$(4.5.1) \quad \begin{array}{ccccc} \mathcal{H}_{\tau} & \xrightarrow{\psi_{\tau}^*} & \mathcal{H}'_{\tau} & \xrightarrow{\phi_{\tau}^*} & \mathcal{H}_{\tau} \\ \downarrow \text{Ha}_{\tau} & & \downarrow \text{Ha}'_{\tau} & & \downarrow \text{Ha}_{\tau} \\ (\mathcal{H}_{\theta^{-1}\tau})(p) & \xrightarrow{\psi_{\theta^{-1}\tau}^*} & (\mathcal{H}'_{\theta^{-1}\tau})(p) & \xrightarrow{\phi_{\theta^{-1}\tau}^*} & (\mathcal{H}_{\theta^{-1}\tau})(p). \end{array}$$

Here and later, if  $\tau = \tau_a$  with  $a \not\equiv 1 \pmod{e}$ , we loose all the Frobenius twists on the modules at  $\theta^{-1}\tau$ . The condition  $\theta^{-1}\tau \in \mathbf{S}$  implies that  $\text{Ker}(\phi_{\theta^{-1}\tau}^*) = \dot{\omega}_{A', \theta^{-1}\tau}^{\otimes p}$  (which would be  $\dot{\omega}_{A', \theta^{-1}\tau}$  if  $\tau = \tau_a$  with  $a \equiv 1 \pmod{e}$ , as said above). Since the image of  $\text{Ha}'_{\tau} : \mathcal{H}'_{\tau} \rightarrow \mathcal{H}_{\theta^{-1}\tau}^{(p)}$  is exactly  $\dot{\omega}_{A', \theta^{-1}\tau}^{\otimes p}$ , we see that the composition  $\phi_{\theta^{-1}\tau}^* \circ \text{Ha}'_{\tau}$  is the zero map, so is the composition  $\text{Ha}_{\tau} \circ \phi_{\tau}^*$ . In particular, this means that  $\text{Im}(\phi_{\tau}^*) \subseteq \text{Ker}(\text{Ha}_{\tau})$ . But  $\tau \in \mathbf{S}$  implies that  $\dot{\omega}_{A, \tau} \subseteq \text{Ker}(\psi_{\tau}^*) = \text{Im}(\phi_{\tau}^*)$ . This shows that  $\text{Ha}_{\tau}(\dot{\omega}_{A, \tau}) = 0$ , meaning that the image of this  $S$ -point under  $\pi_1$  lies in  $X_{\mathbf{T}}$ . Applying this to each  $\tau \in \mathbf{T}$  shows that  $\pi_1(Y_{\mathbf{S}, \mathbf{S}^c}) \subseteq X_{\mathbf{T}}$ .

We now construct the map morphism  $g$  that factors the morphism  $\pi_1$ . Given an  $S$ -point  $y$  of  $Y_{\mathbf{S}, \mathbf{S}^c}$ ,  $\pi_1(y)$  belongs to  $X_{\mathbf{T}}$  as shown above. For each  $\tau \in \mathbf{S} \setminus \theta(\mathbf{S}) = \{\tau_{a_1-\lambda_1}, \tau_{a_2-\lambda_2}, \dots, \tau_{a_r-\lambda_r}\}$ , the image  $\phi_{\tau}^*(\mathcal{H}'_{\tau}) \subseteq \mathcal{H}_{\tau}$  defines a rank one subbundle of the latter. So this lifts the map  $\pi_1$  to a map

$$g : Y_{\mathbf{S}, \mathbf{S}^c} \longrightarrow \mathbb{P}(\mathcal{H}_{\tau_{a_1-\lambda_1}}) \times_{X_{\mathbf{T}}} \cdots \times_{X_{\mathbf{T}}} \mathbb{P}(\mathcal{H}_{\tau_{a_r-\lambda_r}}) =: Y'_{\mathbf{S}, \mathbf{S}^c}.$$

Since  $Y_{\mathbf{S}, \mathbf{S}^c}$  is reduced and  $Y'_{\mathbf{S}, \mathbf{S}^c}$  is normal (as both are smooth over  $\mathbb{F}$ ), we may use the criterion of Frobenius factors as recalled in 4.2. To prove the Proposition, it suffices to show that  $g$  induces a bijection on  $k$ -points for any algebraically closed field  $k$  containing  $\mathbb{F}$ .

We start with a  $k$ -point  $x = (A = \mathcal{A}_x, \lambda, i, \mathcal{F} = \mathcal{F}_x)$  of  $X_{\mathbf{T}} \cap \mathcal{M}_{\mathbf{c}}$ , with  $k$  algebraically closed. We write  $\mathcal{H}_{\tau}$  for  $\mathcal{H}_{\tau, x}$ , and we ignore the Frobenius twist on each  $\mathcal{H}_{\tau}$  ( $\tau \in \Sigma_{\mathbf{p}}$ ), as it is a two-dimensional vector space over an algebraically closed field.

**Claim:** Giving a  $k$ -point  $y$  in  $Y_{\mathbf{S}, \mathbf{S}^c} \cap \pi_1^{-1}(x)$  is equivalent to specifying, for each  $\tau \in \Sigma_{\mathbf{p}}$ , a one-dimensional subspace  $M_{\tau} \subseteq \mathcal{H}_{\tau}$  (which will be the image  $\phi_{\tau}^*(\mathcal{H}'_{\tau})$ ), such that

- (i)  $\text{Ha}_{\tau}(M_{\tau}) \subseteq M_{\theta^{-1}\tau}$ , and
- (ii) for each  $\tau \in \mathbf{S}^c$ ,  $M_{\tau} = \dot{\omega}_{A, \tau}$ , and
- (iii) for each  $\tau \in \theta(\mathbf{S})$ ,  $M_{\tau} = \text{Ker}(\text{Ha}_{\tau})$ .

We temporarily assume this Claim and deduce the Proposition. Note that, for  $\tau \in \mathbf{S}^c \cup \theta(\mathbf{S})$ ,  $M_{\tau}$  is already uniquely determined by (ii) and (iii). Actually, for  $\tau \in \mathbf{S}^c \cap \theta(\mathbf{S}) = \mathbf{T}$ ,  $M_{\tau}$  is “over-determined” as required to be equal to both  $\dot{\omega}_{A, \tau}$  by (ii) and  $\text{Ker}(\text{Ha}_{\tau})$  by (iii), but these two subspaces of  $\mathcal{H}_{\tau}$  are equal as the Hasse invariant  $\dot{h}_{\tau}$  vanishes at  $x$ . The only unspecified  $M_{\tau}$ ’s are those with  $\tau \in \mathbf{S} \setminus \theta(\mathbf{S})$ , or explicitly with  $\tau \in \{\tau_{a_r - \lambda_r}, \tau_{a_{r-1} - \lambda_{r-1}}, \dots, \tau_{a_1 - \lambda_1}\}$  in terms of the combinatorics of 4.4. Their choices are equivalent to specifying a point in  $Y'_{\mathbf{S}, \mathbf{S}^c}$  over  $\pi_1(y)$ . To conclude, we observe that (i) holds for any such choice of  $M_{\tau}$ .

- If  $\tau \in \theta(\mathbf{S}^c)$ ,  $\text{Ha}_{\tau}(M_{\tau}) \subseteq \text{Im}(\text{Ha}_{\tau}) = \dot{\omega}_{A, \theta^{-1}\tau} = M_{\theta^{-1}\tau}$  by (ii).
- If  $\tau \in \theta(\mathbf{S})$ ,  $\text{Ha}_{\tau}(M_{\tau}) = 0$  by (iii).

This completes the proof of the Proposition assuming the Claim.

Now we turn to the proof of the Claim. First, given a point  $y = ((A, \lambda, i, \mathcal{F}); (A', \lambda', i', \mathcal{F}'); \phi, \psi)$  in  $Y_{\mathbf{S}, \mathbf{S}^c} \cap \pi_1^{-1}(x)$ , we set  $M_{\tau} := \phi_{\tau}^*(\mathcal{H}'_{\tau})$ . Since  $y \in Y_{\mathbf{S}, \mathbf{S}^c}$ , we deduce (ii) and (iii) as follows, which further imply (i) formally as shown above.

- for each  $\tau \in \mathbf{S}^c$ ,  $M_{\tau} := \text{Im}(\phi_{\tau}^*) = \text{Ker}(\psi_{\tau}^*) = \dot{\omega}_{A, \tau}$ , and
  - for each  $\tau \in \theta(\mathbf{S})$ ,  $\dot{\omega}_{\theta^{-1}\tau} = \text{Ker}(\phi_{\theta^{-1}\tau}^*)$ ; so  $\phi_{\theta^{-1}\tau}^* \circ \text{Ha}'_{\tau} = 0$ , or equivalently  $\text{Ha}_{\tau} \circ \phi_{\tau}^* = 0$ .
- From this we see that  $M_{\tau} := \text{Im}(\phi_{\tau}^*) = \text{Ker}(\text{Ha}_{\tau})$ .

Conversely, given  $M_{\tau}$  for  $\tau \in \Sigma_{\mathbf{p}}$  satisfying (i)-(iii), let  $\tilde{\mathcal{D}}(A)$  denote the contravariant Dieudonné module, which decomposes into the direct sum

$$\tilde{\mathcal{D}}(A) = \bigoplus_{i=1}^r \bigoplus_{j=1}^{f_i} \tilde{\mathcal{D}}(A)_{\mathbf{p}_i, j}$$

according to the  $\mathcal{O}_F$ -action, so that each summand  $\tilde{\mathcal{D}}(A)_{\mathbf{p}_i, j}$  is a free module of rank two over  $\mathcal{O}_{F_{\mathbf{p}_i}} \otimes_{W(\mathbb{F}_{\mathbf{p}_i}), \tau_{\mathbf{p}_i, j}} W(k) =: \mathcal{O}_{F_{\mathbf{p}_i}, k}$  (which is a complete discrete valuation ring). In particular,  $\mu_{\mathbf{p}_i, j} : \tilde{\mathcal{D}}(A)_{\mathbf{p}_i, j}/p \cong H_{\text{dR}}^1(A)_{\mathbf{p}_i, j}$  is a  $k[x_i]/(x_i^{e_i})$ -module free of rank two, where the  $x_i$  acts by  $[\varpi_i]$ . In particular, we write  $x$  for the action of  $[\varpi]$  on  $\tilde{\mathcal{D}}(A)_{\mathbf{p}, j}$ .<sup>12</sup> For each  $l = 0, 1, \dots, e$ , we write  $\tilde{\mathcal{F}}_{\mathbf{p}, j}^{(l)}$  for the preimage of  $\mathcal{F}_{\mathbf{p}, j}^{(l)}$  under the map  $\mu_{\mathbf{p}, j}$ , and for  $l = 1, \dots, e$ , write  $\tilde{\mathcal{H}}_{\mathbf{p}, j}^{(l)} := x^{-1} \tilde{\mathcal{F}}_{\mathbf{p}, j}^{(l)}$  so that its image under

$$\mu_{\mathbf{p}, j}^{(l)} : \tilde{\mathcal{D}}(A)_{\mathbf{p}, j} \xrightarrow{\mu_{\mathbf{p}, j}} H_{\text{dR}}^1(A)_{\mathbf{p}, j} \rightarrow H_{\text{dR}}^1(A)_{\mathbf{p}, j} / \mathcal{F}_{\mathbf{p}, j}^{(l-1)}$$

is exactly  $\mathcal{H}_{\mathbf{p}, j}^{(l)}$ . In particular,  $\tilde{\mathcal{H}}_{\mathbf{p}, j}^{(l)}$  contains  $\tilde{\mathcal{F}}_{\mathbf{p}, j}^{(l)} = x \tilde{\mathcal{H}}_{\mathbf{p}, j}^{(l-1)}$  as  $\mathcal{O}_{F_{\mathbf{p}}, k}$ -modules with colength 1,<sup>13</sup> and  $\tilde{\mathcal{H}}_{\mathbf{p}, j}^{(1)} = x^{-1} p \tilde{\mathcal{D}}(A)_{\mathbf{p}, j}$ .

Now for each  $l = 1, \dots, e$ , we define  $\tilde{\mathcal{H}}_{\mathbf{p}, j}^{(l)}$  to be the preimage of  $M_{\tau} \subseteq \mathcal{H}_{\tau}$  under the above map  $\mu_{\mathbf{p}, j}^{(l)}$ ; it is an  $\mathcal{O}_{F_{\mathbf{p}}, k}$ -submodule of  $\tilde{\mathcal{H}}_{\mathbf{p}, j}^{(l)}$  of colength 1. Consider the  $\mathcal{O}_F$ -stable submodule

<sup>12</sup>This notation conflicts with earlier where  $x$  denotes a point of  $X_{\mathbf{T}}$ , but we think there should be no confusion.

<sup>13</sup>Here and later, we say an inclusion  $M \subseteq N$  of  $\mathcal{O}_{F_{\mathbf{p}}, k}$ -modules have colength  $i$  if  $N/M$  is a successive extension of  $i$  copies of  $\mathcal{O}_{F_{\mathbf{p}}, k}/(\varpi) \cong k$  as an  $\mathcal{O}_{F_{\mathbf{p}}, k}$ -module.



$\tilde{M} = \oplus_{i=1}^r \oplus_{j=1}^{f_i} \tilde{M}_{\mathfrak{p}_i, j}$  of  $\tilde{\mathcal{D}}(A)$  with

$$\tilde{M}_{\mathfrak{p}_i, j} := \begin{cases} \tilde{\mathcal{D}}(A)_{\mathfrak{p}_i, j} & \text{if } \mathfrak{p}_i \neq \mathfrak{p}, \\ p^{-1}x\tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(1)} & \text{if } \mathfrak{p}_i = \mathfrak{p}. \end{cases}$$

We shall show that  $\tilde{M}$  is a Dieudonné submodule of  $\tilde{\mathcal{D}}(A)$ , namely,  $p\tilde{M} \subseteq V\tilde{M} \subseteq \tilde{M}$ . This is clear for the summands with  $\mathfrak{p}_i \neq \mathfrak{p}$ . At  $\mathfrak{p}$ , in fact, we shall prove the following stronger statements.

- (a) For each  $j = 1, \dots, f$  and  $l = 1, \dots, e-1$ , we have an inclusion  $x\tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(l+1)} \subset \tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(l)}$  of colength 1 (which in particular implies that  $\tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(l)} \subset \tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(l+1)}$ ).
- (b) For each  $j = 1, \dots, f$ , we have an inclusion  $x^{1-e}V(\tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(1)}) \subset \tilde{\mathcal{H}}_{\mathfrak{p}, j-1}'^{(e)}$  of colength 1, which in particular implies that  $\tilde{\mathcal{H}}_{\mathfrak{p}, j-1}'^{(e)} \subset p^{-1}V(\tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(1)})$ .<sup>14</sup>

Provided (a) and (b), we deduce the following inclusion

$$V\tilde{M}_{\mathfrak{p}, j} = V(p^{-1}x\tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(1)}) \subset \tilde{\mathcal{H}}_{\mathfrak{p}, j-1}'^{(e)} \subset x^{-1}\tilde{\mathcal{H}}_{\mathfrak{p}, j-1}'^{(e-1)} \subset \dots \subset x^{1-e}\tilde{\mathcal{H}}_{\mathfrak{p}, j-1}'^{(1)} = \tilde{M}_{\mathfrak{p}, j-1},$$

which has total colength  $e$  (accumulating 1 from each inclusion). So in particular,  $p\tilde{M}_{\mathfrak{p}, j-1}$  is also contained in  $V\tilde{M}_{\mathfrak{p}, j}$ . We now check (a) and (b). For (a), we first note that the multiplication by  $x$  map:  $\cdot x : \tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(l+1)} \rightarrow \tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(l)}$  is injective and its reduction modulo  $x$  is exactly the partial Hasse invariant  $\text{Ha}_{\tau_{\mathfrak{p}, j}^{l+1}} : \mathcal{H}_{\mathfrak{p}, j}^{(l+1)} \rightarrow \mathcal{H}_{\mathfrak{p}, j}^{(l)}$ . The fact  $\text{Ha}_{\tau_{\mathfrak{p}, j}^{l+1}}(M_{\tau_{\mathfrak{p}, j}^{l+1}}) \subseteq M_{\tau_{\mathfrak{p}, j}^l}$  from (i) immediately implies that  $\cdot x$  sends  $\tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(l+1)}$  into  $\tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(l)}$ , where the cokernel can be easily computed to have length 1. To see (b), we first claim that the map  $x^{1-e}V$  takes  $\tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(1)}$  into  $\tilde{\mathcal{H}}_{\mathfrak{p}, j-1}'^{(e)}$ . But this clear, as

$$x^{1-e}V(\tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(1)}) = x^{1-e}V(x^{-1}p\tilde{\mathcal{D}}(A)_{\mathfrak{p}, j}) = \tilde{\omega}_{A, \mathfrak{p}, j-1},$$

and it is an  $\mathcal{O}_{F_{\mathfrak{p}}, k}$ -submodule of  $\tilde{\mathcal{H}}_{\mathfrak{p}, j-1}'^{(e)}$  of colength 1. Moreover, inspecting the construction of  $\text{Hasse}_{\varpi, j}$  recalled in 2.6, we see that the reduction of the map  $x^{1-e}V$  modulo  $x$  is exactly  $\text{Hasse}_{\varpi, j}$ . The condition (i)  $\text{Hasse}_{\varpi, j}(M_{\tau_{\mathfrak{p}, j}^1}) \subseteq M_{\tau_{\mathfrak{p}, j}^e}$  in turn implies that  $x^{1-e}V$  takes  $\tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(1)}$  into  $\tilde{\mathcal{H}}_{\mathfrak{p}, j-1}'^{(e)}$ , and the cokernel of this map is one-dimensional over  $k$ . This completes checking (a) and (b).

Now by standard Dieudonné theory, the inclusions  $p\tilde{M} \subseteq V\tilde{M} \subseteq \tilde{M}$  we just proved above implies that there exists an abelian variety  $A'$  together with  $\mathcal{O}_F$ -equivariant isogenies  $\phi : A \rightarrow A'$  such that the induced map on Dieudonné modules  $\phi^* : \tilde{\mathcal{D}}(A') \rightarrow \tilde{\mathcal{D}}(A)$  can be identified with the natural inclusion  $\tilde{M} \subseteq \tilde{\mathcal{D}}(A)$ . Since by construction  $\tilde{\mathcal{D}}(A)/\tilde{M}$  is a free module of rank one over  $\mathbb{F}_{\mathfrak{p}} \otimes_{\mathbb{F}_p} k$  (as opposed to  $\mathbb{F}_{\mathfrak{p}}[x]/(x^e) \otimes_{\mathbb{F}_p} k$ <sup>15</sup>), we see that there exists a (dual) isogeny  $\psi : A' \rightarrow A$  and a polarization  $\lambda' : A'^{\vee} \rightarrow A' \otimes_{\mathcal{O}_F} \mathfrak{c}'$  satisfying condition (3)(a)-(d) in 3.1. The tame level structure  $i$  on  $A$  naturally propagate to  $A'$ . So it suffices to define a filtration  $\mathcal{F}'$  on  $\omega_{A'}$  satisfying (3)(e) of (3.1) and check that the point defined lies on  $Y_{\text{S}, \text{Sc}}$ . If  $\mathfrak{p}_i \neq \mathfrak{p}$ ,  $\phi^*$  and  $\psi^*$  induce isomorphisms between  $H_{\text{dR}}^1(A)_{\mathfrak{p}_i, j}$  and  $H_{\text{dR}}^1(A')_{\mathfrak{p}_i, j}$ ; this forces  $\mathcal{F}_{\mathfrak{p}_i, j}'^{(l)} = \phi^*(\mathcal{F}_{\mathfrak{p}_i, j}^{(l)})$ . We now consider the case at  $\mathfrak{p}$ . For  $j = 1, \dots, f$  and  $l = 1, \dots, e-1$ , we set

$$\mathcal{F}_{\mathfrak{p}, j}'^{(l)} := x\tilde{\mathcal{H}}_{\mathfrak{p}, j}'^{(l+1)}/p\tilde{\mathcal{D}}(A')_{\mathfrak{p}, j} \quad \text{and} \quad \mathcal{F}_{\mathfrak{p}, j}'^{(e)} := V\tilde{\mathcal{D}}(A')_{\mathfrak{p}, j+1}/p\tilde{\mathcal{D}}(A')_{\mathfrak{p}, j} = \omega_{A', \mathfrak{p}, j};$$

<sup>14</sup>Note that  $x^e$  and  $p$  defines the same ideal in  $\mathcal{O}_{F_{\mathfrak{p}}, k}$ .

<sup>15</sup>The latter condition will give rise to an isogeny such that  $\phi \circ \psi$  is multiplication by  $p$ , but not “multiplication by the ideal  $\mathfrak{p}$ ”.

they are subspaces of  $H_{\text{dR}}^1(A')_{\mathfrak{p},j}$ . Note that by (a) above, we have an inclusion  $\mathcal{F}_{\mathfrak{p},j}^{(l)} \subseteq \mathcal{F}_{\mathfrak{p},j}^{(l+1)}$  of colength 1, for  $l = 1, \dots, e-1$ . Similarly, by (b) above, we have an inclusion

$$x\tilde{\mathcal{H}}_{\mathfrak{p},j}^{(e)} \subset xp^{-1}V(\tilde{\mathcal{H}}_{\mathfrak{p},j}^{(1)}) = xp^{-1}V(px^{-1}\tilde{M}_{\mathfrak{p},j}) = V\tilde{D}(A')_{\mathfrak{p},j+1},$$

which has colength 1. In other words, we have an inclusion  $\mathcal{F}_{\mathfrak{p},j}^{(e-1)} \subset \mathcal{F}_{\mathfrak{p},j}^{(e)}$  of colength 1. All these imply that  $\mathcal{F}_{\mathfrak{p},j}^{(l)}$ 's define the needed filtration  $\underline{\mathcal{F}}$  on  $\omega_{A'}$ . Analogous to the situation on  $A$ , for  $j = 1, \dots, f$  and  $l = 1, \dots, e$ , we set  $\dot{\omega}_{\tau,j}^{(l)} := \mathcal{F}_{\mathfrak{p},j}^{(l)} / \mathcal{F}_{\mathfrak{p},j}^{(l-1)}$ .

Finally, we check that the point we constructed belongs to  $Y_{\mathbb{S},\text{sc}}$ . For this, it is enough to check the following.

- For  $\tau \in \mathbb{S}^c$ ,  $\dot{\omega}_{\tau} = \text{Ker}(\psi_{\tau}^*) = \text{Im}(\phi_{\tau}^*)$ , which follows from the construction and condition (ii).
- For  $\tau \in \mathbb{S}$ ,  $\dot{\omega}_{\tau} = \text{Ker}(\phi_{\tau}^*)$ , which is equivalent to  $\text{Ha}_{\theta\tau} \circ \phi_{\theta\tau}^* = 0$ , which further follows from the construction and condition (iii).

This concludes the verification of the Claim and hence completes the proof of the Proposition.  $\square$

**Corollary 4.6.** *Let  $\mathbb{T}$  be a subset of  $\Sigma_{\mathfrak{p}}$  with  $t = \#\mathbb{T}$ .*

- (1) *Over the open stratum  $X_{\mathbb{T}}^{\circ}$ , the fiber dimension of  $\pi_1$  is less than or equal to  $t$ . The equality holds only when  $\mathbb{T}$  is sparse<sup>16</sup>, namely,  $\tau$  and  $\theta\tau$  does not belong to  $\mathbb{T}$  simultaneously for any  $\tau \in \Sigma_{\mathfrak{p}}$ .*
- (2) *Suppose that  $\mathbb{T}$  is a sparse subset of  $\Sigma_{\mathfrak{p}}$ , then the  $t$ -dimensional fibers of  $\pi_1^{-1}(X_{\mathbb{T}})$  are the (not necessarily disjoint) union of  $Y_{\mathbb{S},\text{sc}}$  such that  $\mathbb{T} = \theta(\mathbb{S}) \setminus \mathbb{S}$ .*
- (3) *We write  $\mathbb{S}_{\underline{\lambda}}$  for the subset  $\mathbb{S}$  given in (4.4.1) as determined by  $\mathbb{T}$  and a tuple  $\underline{\lambda} = (\lambda_1, \dots, \lambda_t)$ . Then for any two tuples  $\underline{\lambda}, \underline{\lambda}'$ , if  $|\lambda'_i - \lambda_i| \geq 2$  for some  $i \in \{1, \dots, r\}$ , then  $\pi_1(Y_{\mathbb{S}_{\underline{\lambda}},\text{sc}} \cap Y_{\mathbb{S}_{\underline{\lambda}',\text{sc}}})$  is disjoint from the open stratum  $X_{\mathbb{T}}^{\circ}$ .*
- (4) *Let  $\underline{\lambda}$  and  $\underline{\lambda}'$  be two tuples such that  $\lambda'_i = \lambda_i$  for  $i \neq i_0$  and  $\lambda'_{i_0} = \lambda_{i_0} + 1$ . Then the ideal sheaf defined by the inclusion  $Y_{\mathbb{S}_{\underline{\lambda}},\text{sc}} \cap Y_{\mathbb{S}_{\underline{\lambda}',\text{sc}}} \hookrightarrow Y_{\mathbb{S}_{\underline{\lambda}},\text{sc}}$  is*

$$\dot{\omega}_{\mathcal{A}',\tau_{a_i-\lambda'_i}} \otimes \dot{\omega}_{\mathcal{A},\tau_{a_i-\lambda_i}}^{-1}.$$

*Proof.* (1) By Proposition 4.5, each irreducible component  $Y_{\mathbb{S},\text{sc}}$  is a fiber bundle of pure dimension  $\#(\theta(\mathbb{S}) \setminus \mathbb{S})$  over  $X_{\theta(\mathbb{S}) \setminus \mathbb{S}}$ ; the base is the union of all open stratum  $X_{\mathbb{T}'}^{\circ}$  with  $\mathbb{T}' \supseteq \theta(\mathbb{S}) \setminus \mathbb{S}$ . In particular, the codimension of  $X_{\mathbb{T}'}^{\circ}$  is  $\#\mathbb{T}'$  which is greater than or equal to the fiber dimension  $\#\theta(\mathbb{S}) \setminus \mathbb{S}$  and the equality holds exactly when  $\mathbb{T}' = \theta(\mathbb{S}) \setminus \mathbb{S}$ . Since this holds true for all irreducible components, (1) is clear.

(2) This is an immediate corollary of (1).

(3) We know that  $Y_{\mathbb{S}_{\underline{\lambda}},\text{sc}} \cap Y_{\mathbb{S}_{\underline{\lambda}',\text{sc}}} = Y_{\mathbb{S}_{\underline{\lambda} \cup \underline{\lambda}',\text{sc}}}$ . If  $\lambda'_i - \lambda_i \geq 2$  for some  $i$ , then this intersection is contained in  $Y_{\mathbb{S}',\text{sc}}$  for

$$\mathbb{S}' = \mathbb{S}_{\underline{\lambda}} \cup \{\tau_{a_i-\lambda_i-2}\}.$$

But then  $\theta(\mathbb{S}') \setminus \mathbb{S}' = \mathbb{T} \cup \{\tau_{a_i-\lambda_i-1}\}$ . So we have

$$\pi_1(Y_{\mathbb{S}_{\underline{\lambda}},\text{sc}} \cap Y_{\mathbb{S}_{\underline{\lambda}',\text{sc}}}) = \pi_1(Y_{\mathbb{S}_{\underline{\lambda} \cup \underline{\lambda}',\text{sc}}}) \subseteq \pi_1(Y_{\mathbb{S}',\text{sc}}) \subseteq X_{\mathbb{T} \cup \{\tau_{a_i-\lambda_i-1}\}}.$$

In particular,  $\pi_1(Y_{\mathbb{S}_{\underline{\lambda}},\text{sc}} \cap Y_{\mathbb{S}_{\underline{\lambda}',\text{sc}}})$  does not intersect with  $X_{\mathbb{T}}^{\circ}$ .

(4) Note that the condition implies that  $\mathbb{S}_{\underline{\lambda}'} = \mathbb{S}_{\underline{\lambda}} \cup \{\tau_{a_i-\lambda'_i}\}$ . So  $Y_{\mathbb{S}_{\underline{\lambda}},\text{sc}} \cap Y_{\mathbb{S}_{\underline{\lambda}',\text{sc}}} = Y_{\mathbb{S}_{\underline{\lambda}',\text{sc}}}$  is a closed subscheme in  $Y_{\mathbb{S}_{\underline{\lambda}},\text{sc}}$  defined as the vanishing locus of

$$\phi_{\tau_{a_i-\lambda'_i}}^* : \dot{\omega}_{\mathcal{A}',\tau_{a_i-\lambda'_i}} \longrightarrow \dot{\omega}_{\mathcal{A},\tau_{a_i-\lambda'_i}}.$$

<sup>16</sup>This is called *spaced* in [GKa12].

The statement of (4) follows, because the ideal sheaf of the vanishing locus of a section of a line bundle is the inverse line bundle.  $\square$

Before proceeding, we need some additional geometric information regarding the map  $\pi_1 : Y_{\mathbf{S}, \mathbf{S}^c} \rightarrow X_{\mathbf{T}}$ .

**Proposition 4.7.** *Let  $\mathbf{S}$  be a subset of  $\Sigma_p$  and put  $\mathbf{T} := \theta(\mathbf{S}) \setminus \mathbf{S}$ .*

(1) *We have the following isomorphisms of line bundles on  $Y_{\mathbf{S}, \mathbf{S}^c}$ :*

$$(4.7.1) \quad \text{if } \tau, \theta^{-1}\tau \in \mathbf{S}, \quad \dot{\omega}_{\mathcal{A}', \tau} \cong \begin{cases} \dot{\omega}_{\mathcal{A}, \theta^{-1}\tau}^{\otimes p} & \text{if } \tau = \tau_a \text{ with } a \equiv 1 \pmod{e}, \\ \dot{\omega}_{\mathcal{A}, \theta^{-1}\tau} & \text{otherwise,} \end{cases}$$

$$(4.7.2) \quad \text{if } \tau, \theta^{-1}\tau \in \mathbf{S}^c, \quad \dot{\omega}_{\mathcal{A}, \tau} \cong \begin{cases} \dot{\omega}_{\mathcal{A}', \theta^{-1}\tau}^{\otimes p} & \text{if } \tau = \tau_a \text{ with } a \equiv 1 \pmod{e}, \\ \dot{\omega}_{\mathcal{A}', \theta^{-1}\tau} & \text{otherwise.} \end{cases}$$

(2) *For each  $\tau \in \mathbf{S} \setminus \theta(\mathbf{S})$ , we write  $\mathcal{O}_{\tau}(-1)$  for the canonical sub line bundle on  $Y'_{\mathbf{S}, \mathbf{S}^c}$  at the  $\mathbb{P}^1$ -factor indexed by  $\tau$ . Then we have*

$$\dot{\omega}_{\mathcal{A}', \tau_{a_i - \lambda_i}} \cong g^* \mathcal{O}_{\tau_{a_i - \lambda_i}}(1), \quad \text{and} \quad g^* \mathcal{O}_{\tau_{a_i - \lambda_i}}(-1) \cong \begin{cases} \dot{\omega}_{\mathcal{A}', \tau_{a_i - \lambda_i} - 1}^{\otimes p} & \text{if } a_i - \lambda_i \equiv 1 \pmod{e} \\ \dot{\omega}_{\mathcal{A}', \tau_{a_i - \lambda_i} - 1} & \text{otherwise.} \end{cases}.$$

*Proof.* (1) We shall prove (4.7.2) and the proof of (4.7.1) is similar. So we assume that  $\tau, \theta^{-1}\tau \in \mathbf{S}^c$ . For simplicity, we assume that  $\tau = \tau_a$  with  $a \equiv 1 \pmod{e}$ ; the argument for the other case is similar by loosing all the Frobenius twists in the proof (and hence getting  $\dot{\omega}_{\mathcal{A}', \theta^{-1}\tau}$  as opposed to  $\dot{\omega}_{\mathcal{A}', \theta^{-1}\tau}^{\otimes p}$  on the right hand side of (4.7.2)). Take an  $S$ -point of  $Y_{\mathbf{S}, \mathbf{S}^c}$ ; we look at the commutative diagram (4.5.1) which we copy to below

$$(4.7.3) \quad \begin{array}{ccccc} \mathcal{H}_{\tau} & \xrightarrow{\psi_{\tau}^*} & \mathcal{H}'_{\tau} & \xrightarrow{\phi_{\tau}^*} & \mathcal{H}_{\tau} \\ \downarrow \text{Ha}_{\tau} & & \downarrow \text{Ha}'_{\tau} & & \downarrow \text{Ha}_{\tau} \\ (\mathcal{H}_{\theta^{-1}\tau})^{(p)} & \xrightarrow{\psi_{\theta^{-1}\tau}^*} & (\mathcal{H}'_{\theta^{-1}\tau})^{(p)} & \xrightarrow{\phi_{\theta^{-1}\tau}^*} & (\mathcal{H}_{\theta^{-1}\tau})^{(p)}. \end{array}$$

Since  $\tau, \theta^{-1}\tau \in \mathbf{S}^c$ , we have

$$\dot{\omega}_{\mathcal{A}, \tau} \cong \text{Ker}(\psi_{\tau}^* : \mathcal{H}_{\tau} \rightarrow \mathcal{H}'_{\tau}) = \text{Im}(\phi_{\tau}^* : \mathcal{H}'_{\tau} \rightarrow \mathcal{H}_{\tau}).$$

Note that  $\dot{\omega}_{\mathcal{A}', \theta^{-1}\tau}^{\otimes p}$  is also the image of  $\mathcal{H}'_{\tau}$  but under the map  $\text{Ha}'_{\tau}$ . To prove the desired isomorphism, it suffices to show that

$$(4.7.4) \quad \text{Ker}(\text{Ha}'_{\tau}) = \text{Ker}(\phi_{\tau}^*) = \text{Im}(\psi_{\tau}^*).$$

Since both sides are subbundle of  $\mathcal{H}'_{\tau}$  of rank one, it suffices to show that  $\text{Ha}'_{\tau} \circ \psi_{\tau}^* = 0$ , which is equivalent to show that  $\psi_{\theta^{-1}\tau}^* \circ \text{Ha}_{\tau} = 0$ . But the image of  $\text{Ha}_{\tau}$  is exactly  $\dot{\omega}_{\mathcal{A}, \theta^{-1}\tau}^{\otimes p}$  which lies in the kernel of  $\psi_{\theta^{-1}\tau}^*$  by the assumption  $\theta^{-1}\tau \in \mathbf{S}^c$ . So we conclude (4.7.4) and hence prove (1).

(2) The first equality follows from the equality

$$\wedge^2 \mathcal{H}'_{\tau_{a_i - \lambda_i}} \cong \text{Ker} \phi_{\tau_{a_i - \lambda_i}}^* \otimes \text{Im} \phi_{\tau_{a_i - \lambda_i}}^* \cong \dot{\omega}_{\mathcal{A}', \tau_{a_i - \lambda_i}} \otimes g^* \mathcal{O}_{\tau_{a_i - \lambda_i}}(-1),$$

because the left hand side  $\wedge^2 \mathcal{H}'_{\tau_{a_i - \lambda_i}}$  can be canonically trivialized over  $\mathcal{M}(\mathfrak{p})$  by 4.1(3) and [RX14<sup>+</sup>, Lemma 2.5] (through pulling back along  $\pi_2$ ).

For the second equality, we shall only prove it when  $a_i - \lambda_i \equiv 1 \pmod{e}$ ; the other case is similar but without the additional Frobenius pullbacks. We take an  $S$ -point of  $Y_{\mathbf{S}, \mathbf{S}^c}$  as above; we may look

at (4.7.3) for  $\tau = \tau_{a_i - \lambda_i}$ . We note that  $\dot{\omega}_{A', \tau_{a_i - \lambda_i - 1}}^{\otimes p}$  is the image of  $\mathcal{H}'_{\tau_{a_i - \lambda_i}}$  under  $\text{Ha}'_{\tau_{a_i - \lambda_i}}$ , and  $g^* \mathcal{O}_{\tau_{a_i - \lambda_i}}(-1)$  is the image of  $\mathcal{H}'_{\tau_{a_i - \lambda_i}}$  under  $\phi_{\tau_{a_i - \lambda_i}}^*$ . So it suffices to prove that

$$\text{Ker}(\text{Ha}'_{\tau_{a_i - \lambda_i}}) \cong \text{Ker}(\phi_{\tau_{a_i - \lambda_i}}^*) = \text{Im}(\psi_{\tau_{a_i - \lambda_i}}^*).$$

Similar to the argument in (1), for rank reasons, it suffices to show that

$$\text{Ha}'_{\tau_{a_i - \lambda_i}} \circ \psi_{\tau_{a_i - \lambda_i}}^* : \mathcal{H}_{\tau_{a_i - \lambda_i}} \longrightarrow (\mathcal{H}'_{\tau_{a_i - \lambda_i - 1}})^{(p)}$$

is the zero map. But this follows from that  $\psi_{\tau_{a_i - \lambda_i - 1}}^*(\dot{\omega}_{A, \tau_{a_i - \lambda_i - 1}}^{(p)}) = 0$  because  $\tau_{a_i - \lambda_i - 1} \in \mathbf{S}^c$ .  $\square$

The following corollary of Grothendieck's formal function theorem will reduce Proposition 3.18 to a calculation at the component described in Corollary 4.6(2).

**Proposition 4.8.** *Let  $h : X \rightarrow Y$  be a projective morphism between noetherian schemes, and let  $t = \max\{\dim X_y | y \in Y\}$ . Let  $\mathcal{F}$  be a coherent sheaf on  $X$ .*

(1) *Then  $R^i h_*(\mathcal{F}) = 0$  for all  $i > t$ .*

(2) *Suppose that  $X$  is the union of two components  $X_1 \cup X_2$ , such that  $\max\{\dim X_{2,y} | y \in Y\} < t$ . Then we have*

$$R^t h_*(\mathcal{F}) \cong R^t h_*(\mathcal{F}|_{X_1}).$$

*Proof.* (1) is a corollary of Grothendieck's formal function theorem; see e.g. [Hartshorne, Cor 11.2].

(2) Write  $i : X_1 \rightarrow X$  for the natural inclusion. Let  $\mathcal{G}$  denote the kernel of the surjective morphism  $\mathcal{F} \rightarrow i_* \mathcal{F}|_{X_1}$ ; then  $\mathcal{G}$  is supported on  $X_2$ . By (1),  $R^r h_*(\mathcal{G}) = R^{r+1} h_*(\mathcal{G}) = 0$ . So we proved (2).  $\square$

**4.9. Proof of Proposition 3.18.** We are now ready to prove Proposition 3.18.

By Corollary 4.6(1) and Proposition 4.8(1), it suffices to show that, for each sparse set  $\mathbf{T} \subseteq \Sigma_{\mathbf{p}}$  with  $t = \#\mathbf{T}$ ,  $R^t \pi_{1,*} \pi_2^* \dot{\omega}_{\mathbb{F}}^{\kappa}$  vanishes on every *geometric* generic point  $\eta_{\mathbf{T}}$  of  $X_{\mathbf{T}}^{\circ}$ . By Corollary 4.6(2), the  $t$ -dimension fibers of  $X_{\mathbf{T}}$  are exactly those of  $Y_{\mathbf{S}, \mathbf{S}^c}$  for which  $\theta(\mathbf{S}) \setminus \mathbf{S} = \mathbf{T}$ . Write  $Z_{\mathbf{T}}$  for the union of these  $Y_{\mathbf{S}, \mathbf{S}^c}$  (with the reduced scheme structure). Using Proposition 4.8(2), we see that

$$(R^r \pi_{1,*} \pi_2^* \dot{\omega}_{\mathbb{F}}^{\kappa})_{\eta_{\mathbf{T}}} \cong (R^r \pi_{1,*} (\pi_2^* \dot{\omega}_{\mathbb{F}}^{\kappa})|_{Z_{\mathbf{T}}})_{\eta_{\mathbf{T}}}.$$

Here and after, we shall frequently write  $(-)\eta_{\mathbf{T}}$  to indicate the base change to the point  $\eta_{\mathbf{T}}$ . We shall prove Proposition 3.18 in the following two steps:

- (1) Let  $(Y_{\mathbf{S}, \mathbf{S}^c})_{\eta_{\mathbf{T}}}^{\text{red}}$  denote the reduced subscheme of  $(Y_{\mathbf{S}, \mathbf{S}^c})_{\eta_{\mathbf{T}}}$ . The natural map  $g_{\eta_{\mathbf{T}}}^{\text{red}} : (Y_{\mathbf{S}, \mathbf{S}^c})_{\eta_{\mathbf{T}}}^{\text{red}} \rightarrow (Y'_{\mathbf{S}, \mathbf{S}^c})_{\eta_{\mathbf{T}}} \cong (\mathbb{P}^1)_{\eta_{\mathbf{T}}}^t$  is the map that is the  $p$ -Frobenius in the factor labeled by  $\tau$  for which  $\tau = \tau_a$  with  $a \equiv 1 \pmod{e}$ ; so in particular,  $(Y_{\mathbf{S}, \mathbf{S}^c})_{\eta_{\mathbf{T}}}^{\text{red}}$  itself is isomorphic to  $(\mathbb{P}^1)_{\eta_{\mathbf{T}}}^t$ .
- (2)  $(\pi_2^* \dot{\omega}_{\mathbb{F}}^{\kappa})_{(Z_{\mathbf{T}})_{\eta_{\mathbf{T}}}}$  is a successive extension of line bundles  $L_{\mathbf{S}}$  supported on each  $(Y_{\mathbf{S}, \mathbf{S}^c})_{\eta_{\mathbf{T}}}$ , and  $L_{\mathbf{S}}|_{(Y_{\mathbf{S}, \mathbf{S}^c})_{\eta_{\mathbf{T}}}^{\text{red}}}$  is the external tensor product of line bundles on  $\mathbb{P}_{\eta_{\mathbf{T}}}^1$  of the form  $\mathcal{O}(n)$  with  $n \geq -1$  (assuming our conditions on weights in Proposition 3.18).

We start with (1). Note that the map  $g : Y_{\mathbf{S}, \mathbf{S}^c} \rightarrow Y'_{\mathbf{S}, \mathbf{S}^c}$  is a Frobenius factor, so the base change  $(Y_{\mathbf{S}, \mathbf{S}^c})_{\eta_{\mathbf{T}}}$  to the *geometric* generic point may not be reduced; we write  $(Y_{\mathbf{S}, \mathbf{S}^c})_{\eta_{\mathbf{T}}}^{\text{red}}$  for its reduced subscheme. Then the base change of  $g$  over to  $\eta_{\mathbf{T}}$ , denoted by  $g_{\eta_{\mathbf{T}}}$ , gives a Frobenius factor (over the residue field  $\kappa_{\mathbf{T}}$  at  $\eta_{\mathbf{T}}$ ):

$$g_{\eta_{\mathbf{T}}}^{\text{red}} : (Y_{\mathbf{S}, \mathbf{S}^c})_{\eta_{\mathbf{T}}}^{\text{red}} \rightarrow (Y'_{\mathbf{S}, \mathbf{S}^c})_{\eta_{\mathbf{T}}}.$$

We claim that this is in fact the  $p$ -Frobenius in the factor labeled by  $\tau$  for which  $\tau = \tau_a$  with  $a \equiv 1 \pmod{e}$ , and isomorphism on other factors.

To ease the presentation, we may extend both  $\mathcal{M}$  and  $\mathcal{M}(\mathbf{p})$  from over  $\mathcal{O}$  to over the completion of maximal unramified extension of  $\mathcal{O}$ . This way, all closed points of  $\mathcal{M}$  and  $\mathcal{M}(\mathbf{p})$  are defined over  $\overline{\mathbb{F}}_p$ . We follow the proof of Proposition 3.3 to take a small enough Zariski open neighborhood  $\mathcal{U} \subset \mathcal{M}$

of  $\eta_{\mathbf{T}}$  (in the integral model) and then take a small enough Zariski open subset  $\mathcal{V} \subset \pi_1^{-1}(\mathcal{U}) \subset \mathcal{M}(\mathfrak{p})$  intersecting the fiber  $Y_{\mathbf{S}, \mathbf{S}^c}$ , such that the tuple

$$(\mathcal{H}_{\tau}|_{\mathcal{V}}, \mathcal{H}'_{\tau}|_{\mathcal{V}}, \phi_{\tau}^*, \psi_{\tau}^*)_{\tau \in \Sigma_{\mathfrak{p}}} \quad \text{is isomorphic to} \quad (\mathcal{O}_{\mathcal{V}}^{\oplus 2}, \mathcal{O}_{\mathcal{V}}^{\oplus 2}, \begin{pmatrix} 1 & 0 \\ 0 & \tau(\varpi) \end{pmatrix}, \begin{pmatrix} \tau(\varpi) & 0 \\ 0 & 1 \end{pmatrix})_{\tau \in \Sigma_{\mathfrak{p}}}.$$

Let  $\mathbf{F}$  denote the moduli problem of rank one  $\mathcal{O}_{\mathcal{V}}$ -subbundle  $M_{\tau} \subseteq \mathcal{O}_{\mathcal{V}}^{\oplus 2}$  for each  $\tau \in \Sigma_{\mathfrak{p}} \setminus \mathbf{T}$  corresponding to the subbundle  $\dot{\omega}_{\mathcal{A}, \tau}|_{\mathcal{V}} \subset \mathcal{H}_{\tau}|_{\mathcal{V}}$ . Let  $\mathbf{G}$  denote the moduli problem of rank one  $\mathcal{O}_{\mathcal{V}}$ -subbundles  $M_{\tau} \subset \mathcal{O}_{\mathcal{V}}^{\oplus 2}$  for each  $\tau \in \mathbf{S}$  corresponding to the subbundles  $\dot{\omega}_{\mathcal{A}, \tau}|_{\mathcal{V}} \subset \mathcal{H}_{\tau}|_{\mathcal{V}}$  and rank one subbundle  $M'_{\tau} \subset \mathcal{O}_{\mathcal{V}}^{\oplus 2}$  for each  $\tau \in \mathbf{S}^c$  corresponding to the subbundle  $\dot{\omega}_{\mathcal{A}', \tau}|_{\mathcal{V}} \subseteq \mathcal{H}'_{\tau}|_{\mathcal{V}}$ . The theory of local model says that  $X_{\mathbf{T}}$  (resp.  $Y_{\mathbf{S}, \mathbf{S}^c}$ ) is étale locally isomorphic to  $\mathbf{F}_{\overline{\mathbb{F}}_p}$  (resp.  $\mathbf{G}_{\overline{\mathbb{F}}_p}$ ). The local parameters of  $\mathbf{F}_{\overline{\mathbb{F}}_p}$  (at a point) are  $u_{\tau}$  for  $\tau \in \Sigma_{\mathfrak{p}} \setminus \mathbf{T}$  which measures the position  $\dot{\omega}_{\mathcal{A}, \tau}|_{\mathcal{V}} \subset \mathcal{H}_{\tau}|_{\mathcal{V}}$ . In particular, the completion of  $X_{\mathbf{T}}$  at a closed  $\overline{\mathbb{F}}_p$ -point  $x$  is isomorphic to  $\overline{\mathbb{F}}_p[[ (u_{\tau})_{\tau \in \Sigma_{\mathfrak{p}} \setminus \mathbf{T}} ]]$ . The local parameters of  $\mathbf{G}_{\overline{\mathbb{F}}_p}$  (at a point) are  $u_{\tau}$  for  $\tau \in \mathbf{S}$  which measures the position of  $\dot{\omega}_{\mathcal{A}, \tau}|_{\mathcal{V}} \subset \mathcal{H}_{\tau}|_{\mathcal{V}}$ , and  $v_{\tau}$  for  $\tau \in \mathbf{S}^c$  which measures the position of  $\dot{\omega}_{\mathcal{A}', \tau}|_{\mathcal{V}} \subset \mathcal{H}'_{\tau}|_{\mathcal{V}}$ . In particular, the completion of  $Y_{\mathbf{S}, \mathbf{S}^c}$  at a closed  $\overline{\mathbb{F}}_p$ -point  $y \in \pi_1^{-1}(x) \cap Y_{\mathbf{S}, \mathbf{S}^c}$  is isomorphic to  $\overline{\mathbb{F}}_p[[ (u_{\tau})_{\tau \in \mathbf{S}}, (v_{\tau})_{\tau \in \mathbf{S}^c} ]]$ . Note that we can use the same notation  $u_{\tau}$  for local parameters on  $\mathbf{F}_{\overline{\mathbb{F}}_p}$  and on  $\mathbf{G}_{\overline{\mathbb{F}}_p}$  because in the homomorphism

$$(4.9.1) \quad \mathcal{O}_{\mathcal{U}, x}^{\wedge} \cong \overline{\mathbb{F}}_p[[ (u_{\tau})_{\tau \in \Sigma_{\mathfrak{p}} \setminus \mathbf{T}} ]] \longrightarrow \mathcal{O}_{\mathcal{V}, y}^{\wedge} \cong \overline{\mathbb{F}}_p[[ (u_{\tau})_{\tau \in \mathbf{S}}, (v_{\tau})_{\tau \in \mathbf{S}^c} ]]$$

on the completions induced by  $\pi_1$ , one may choose the local parameters in a compatible way so that  $u_{\tau}$  for  $\tau \in \mathbf{S}$  is taken to  $u_{\tau}$ .

To understand the image of  $u_{\tau}$  for  $\tau \in \Sigma_{\mathfrak{p}} \setminus \mathbf{T} \setminus \mathbf{S} = \mathbf{S}^c \cap \sigma(\mathbf{S}^c)$ , we consider a variant of the argument of Proposition 4.7(1). If  $\tau, \theta^{-1}\tau \in \mathbf{S}^c$  and if  $\tau = \tau_a$  with  $a \equiv 1 \pmod{e}$ , the proof of Proposition 4.7(1) implies that  $\text{Ker}(\text{Ha}'_{\tau}) = \text{Ker}(\phi_{\tau, \overline{\mathbb{F}}_p}^*)$ . So we may choose an isomorphism  $\eta_{\tau} : \mathcal{H}_{\tau, \overline{\mathbb{F}}_p} \cong \mathcal{H}'_{\theta^{-1}\tau, \overline{\mathbb{F}}_p}$  such that  $\text{Ha}'_{\tau}$  is the same as  $\eta_{\tau} \circ \phi_{\tau, \overline{\mathbb{F}}_p}^*$ . Under this identification, we have

$$\eta_{\tau}(\dot{\omega}_{\mathcal{A}, \tau, \overline{\mathbb{F}}_p}) = \eta_{\tau}(\text{Im}(\phi_{\tau, \overline{\mathbb{F}}_p}^*)) = \text{Im}(\text{Ha}'_{\tau}) = \dot{\omega}_{\mathcal{A}', \theta^{-1}\tau, \overline{\mathbb{F}}_p}^{\otimes p}.$$

So we see that we can rearrange the choices of local parameters so that the local parameter  $u_{\tau}$  (for  $\tau \in \mathbf{S}^c \cap \theta(\mathbf{S}^c)$  and  $\tau = \tau_a$  with  $a \equiv 1 \pmod{e}$ ) is sent to  $v_{\theta^{-1}\tau}^p$  under the map (4.9.1). The same argument shows that, when  $\tau \in \mathbf{S}^c \cap \theta(\mathbf{S}^c)$  and  $\tau = \tau_a$  with  $a \not\equiv 1 \pmod{e}$ , we can rearrange the choices of local parameters so that  $u_{\tau}$  is sent to  $v_{\theta^{-1}\tau}$ .

Using this, we see that the completion at a closed point  $y_{\eta_{\mathbf{T}}}$  of  $(Y_{\mathbf{S}, \mathbf{S}^c})_{\eta_{\mathbf{T}}}$  is isomorphic to

$$\overline{\mathbb{F}}_p((u_{\tau})_{\tau \in \Sigma_{\mathfrak{p}} \setminus \mathbf{T}})^{\text{alg}} \otimes_{\overline{\mathbb{F}}_p[[ (u_{\tau})_{\tau \in \Sigma_{\mathfrak{p}} \setminus \mathbf{T}} ]]} \overline{\mathbb{F}}_p[[ (u_{\tau})_{\tau \in \mathbf{S}}, (v_{\theta^{-1}\tau})_{\tau \in \mathbf{S}^c \cap \theta(\mathbf{S}^c)} ]][[ (v_{\tau})_{\tau \in \theta^{-1}(\mathbf{S}) \cap \mathbf{S}^c} ]].$$

Using the identification of  $u_{\tau}$  with  $v_{\theta^{-1}\tau}^p$  (resp.  $v_{\theta^{-1}\tau}$ ) for  $\tau \in \mathbf{S}^c \cap \theta(\mathbf{S}^c)$  with  $\tau = \tau_a$  for  $a \equiv 1 \pmod{e}$  (resp.  $a \not\equiv 1 \pmod{e}$ ), we see that the completion of  $(Y_{\mathbf{S}, \mathbf{S}^c})_{\eta_{\mathbf{T}}}^{\text{red}}$  at a closed point  $y_{\eta_{\mathbf{T}}}$  is isomorphic to

$$\kappa_{\mathbf{T}}[[ (v_{\tau})_{\tau \in \theta^{-1}(\mathbf{S}) \cap \mathbf{S}^c} ]].$$

Here we recall that  $\kappa_{\mathbf{T}}$  is the residue field of  $\eta_{\mathbf{T}}$  and  $v_{\tau}$  is the coordinate for the subbundle  $\dot{\omega}_{\mathcal{A}', \tau} \subseteq \mathcal{H}'_{\tau}$ .

On the other hand, the completion of  $(Y'_{\mathbf{S}, \mathbf{S}^c})_{\eta_{\mathbf{T}}}$  at  $f_{\eta_{\mathbf{T}}}(y_{\eta_{\mathbf{T}}})$  is isomorphic to  $\kappa_{\mathbf{T}}[[ (v'_{\tau})_{\tau \in \mathbf{S} \setminus \theta(\mathbf{S})} ]]$ , where  $v'_{\tau}$  is the coordinate of the chosen subbundle of  $\mathcal{H}_{\tau}$  in the definition of  $Y'_{\mathbf{S}, \mathbf{S}^c}$ . We need to show that, up to adjusting the local parameter  $v'_{\tau}$ ,

$$(4.9.2) \quad \text{for every } \tau \in \mathbf{S} \setminus \sigma(\mathbf{S}), \quad f_{\eta_{\mathbf{T}}}^*(v'_{\tau}) = \begin{cases} v_{\theta^{-1}\tau}^p & \text{if } \tau = \tau_a \text{ with } a \equiv 1 \pmod{e}; \\ v_{\theta^{-1}\tau} & \text{if } \tau = \tau_a \text{ with } a \not\equiv 1 \pmod{e}. \end{cases}$$

For this, we fix one such  $\tau \in \mathbf{S} \setminus \theta(\mathbf{S})$ . We assume that  $\tau = \tau_a$  with  $a \equiv 1 \pmod{e}$  (and the other case can be proved in the same way by removing all the Frobenius twists). Following exactly the same argument as above, we start by noticing that  $\text{Ker}(\text{Ha}'_{\tau}) = \text{Ker}(\phi_{\tau, \overline{\mathbb{F}}_p}^*)$ . So we may choose an

isomorphism  $\eta_\tau : \mathcal{H}_{\tau, \mathbb{F}_p} \cong \mathcal{H}'_{\theta^{-1}\tau, \mathbb{F}_p}$  such that  $\text{Ha}'_\tau$  is the same as  $\eta_\tau \circ \phi_{\tau, \mathbb{F}_p}^*$ . Under this identification, we have

$$\eta_\tau(\text{Im}(\phi_{\tau, \mathbb{F}_p}^*)) = \text{Im}(\text{Ha}'_\tau) = \dot{\omega}_{\mathcal{A}', \theta^{-1}\tau, \mathbb{F}_p}^{\otimes p}.$$

So it follows that, up to adjusting the local parameter, (4.9.2) holds. Since  $g_{\eta_\tau}^{\text{red}}$  is already a Frobenius factor, it must take the form as described in (1).

Now we may identify  $(Y_{\mathbf{s}, \mathbf{s}^c})_{\eta_\tau}^{\text{red}}$  with  $(\mathbb{P}_{\eta_\tau}^1)^t$ . Write  $\mathcal{O}_i(1)$  for the canonical quotient bundle from the  $i$ th factor. In particular,  $g_{\eta_\tau}^* \mathcal{O}_{\tau_{a_i - \lambda_i}}(1)$  is equal to  $\mathcal{O}_i(p)$  if  $a_i - \lambda_i \equiv 1 \pmod{e}$  and to  $\mathcal{O}_i(1)$  otherwise. As a corollary of this and Proposition 4.7, we have

$$(4.9.3) \quad \dot{\omega}_{\mathcal{A}', \tau} |_{(Y_{\mathbf{s}, \mathbf{s}^c})_{\eta_\tau}^{\text{red}}} \cong \begin{cases} \mathcal{O}_i(p) & \text{if } \tau = \tau_{a_i - \lambda_i} \text{ and } a_i - \lambda_i \equiv 1 \pmod{e}, \\ \mathcal{O}_i(1) & \text{if } \tau = \tau_{a_i - \lambda_i} \text{ and } a_i - \lambda_i \not\equiv 1 \pmod{e}, \\ \mathcal{O}_i(-1) & \text{if } \tau = \tau_{a_i - \lambda_i - 1}, \\ \mathcal{O}_{(Y_{\mathbf{s}, \mathbf{s}^c})_{\eta_\tau}^{\text{red}}} & \text{otherwise.} \end{cases}$$

We now turn to (2). Corollary 4.6(3) and (4) explained the intersection relation among  $Y_{\mathbf{s}_\lambda, \mathbf{s}_\lambda^c}$ 's. Put  $s_i = a_i - a_{i-1} - 1$  for  $i \geq 2$  and  $s_1 = a_1 + ef - a_t - 1$ . For example when  $t = 2$ , the following diagram shows the intersection relation, where two irreducible components of  $Z_\tau$  intersect in codimension 1 if they are linked by a line, and they intersect in codimension 2 if they are at the opposite vertices of a square:

$$\begin{array}{ccccccc} Y_{\mathbf{s}_{1,1}, \mathbf{s}_{1,1}^c} & \text{---} & Y_{\mathbf{s}_{1,2}, \mathbf{s}_{1,2}^c} & \text{---} & \cdots & \text{---} & Y_{\mathbf{s}_{1,s_2}, \mathbf{s}_{1,s_2}^c} \\ | & & | & & & & | \\ Y_{\mathbf{s}_{2,1}, \mathbf{s}_{2,1}^c} & \text{---} & Y_{\mathbf{s}_{2,2}, \mathbf{s}_{2,2}^c} & \text{---} & \cdots & \text{---} & Y_{\mathbf{s}_{2,s_2}, \mathbf{s}_{2,s_2}^c} \\ | & & | & & \ddots & & | \\ \vdots & & \vdots & & & & \vdots \\ | & & | & & & & | \\ Y_{\mathbf{s}_{s_1,1}, \mathbf{s}_{s_1,1}^c} & \text{---} & Y_{\mathbf{s}_{s_1,2}, \mathbf{s}_{s_1,2}^c} & \text{---} & \cdots & \text{---} & Y_{\mathbf{s}_{s_1,s_2}, \mathbf{s}_{s_1,s_2}^c} \end{array}$$

Moreover, these  $Y_{\mathbf{s}_\lambda, \mathbf{s}_\lambda^c}$ 's have proper intersections by the proof of Proposition 3.3. So by Corollary 4.6(4),  $\dot{\omega}^\kappa|_{Z_\tau}$  is the successive extension of

$$(4.9.4) \quad \dot{\omega}^\kappa|_{Y_{\mathbf{s}_\lambda, \mathbf{s}_\lambda^c}} \otimes \bigotimes_{i=1: \lambda_i \neq s_i}^t (\dot{\omega}_{\mathcal{A}', \tau_{a_i - \lambda_i - 1}} \otimes \dot{\omega}_{\mathcal{A}, \tau_{a_i - \lambda_i - 1}}^{-1}) \quad \text{for all } \lambda_i \in \{1, \dots, s_i\},$$

in which the term with  $\underline{\lambda} = \underline{1}$  is the subobject and the term with  $\underline{\lambda} = (s_i)_{i=1, \dots, t}$  is the quotient object. Restricting this to the  $(\mathbb{P}_{\eta_\tau}^1)^t$ -bundle  $(Y'_{\mathbf{s}_\lambda, \mathbf{s}_\lambda^c})_{\eta_\tau}^{\text{red}}$ , this is equal to

$$\bigotimes_{i=1}^t \begin{cases} \mathcal{O}_i(pk_{a_i - \lambda_i} - k_{a_i - \lambda_i - 1}) & \text{if } \lambda_i = s_i \text{ and } a_i - \lambda_i \equiv 1 \pmod{e}, \\ \mathcal{O}_i(k_{a_i - \lambda_i} - k_{a_i - \lambda_i - 1}) & \text{if } \lambda_i = s_i \text{ and } a_i - \lambda_i \not\equiv 1 \pmod{e}, \\ \mathcal{O}_i(pk_{a_i - \lambda_i} - k_{a_i - \lambda_i - 1} - 1) & \text{if } \lambda_i \neq s_i \text{ and } a_i - \lambda_i \equiv 1 \pmod{e}, \\ \mathcal{O}_i(k_{a_i - \lambda_i} - k_{a_i - \lambda_i - 1} - 1) & \text{if } \lambda_i \neq s_i \text{ and } a_i - \lambda_i \not\equiv 1 \pmod{e}. \end{cases}$$

By the assumption of Proposition 3.18, the numbers in the parentheses of the right hand side are always  $\geq 1$ . Since  $H^1(\mathbb{P}^1, \mathcal{O}(n)) = 0$  for  $n \geq -1$ , we see that

$$R^{>0}\pi_{1,*} \left( (4.9.4)|_{(Y'_{\mathbf{s}_\lambda, \mathbf{s}_\lambda^c})_{\eta_\tau}^{\text{red}}} \right) = 0.$$

It then follows that

$$(4.9.5) \quad R^t \pi_{1,*}((\pi_2^* \dot{\omega}^\kappa)|_{(Z_T)_{\eta_T}^{\text{red}}}) = 0.$$

Due the cohomological dimension, by Proposition 4.8(1),  $R^t \pi_{1,*}(-)$  is a *right exact* functor on sheaves set-theoretically supported on  $(Z_T)_{\eta_T}^{\text{red}}$ , and it is trivial on any coherent sheaf set-theoretically supported in dimension  $< t$  subspace of  $(Z_T)_{\eta_T}^{\text{red}}$ . We show below that, by (a variant of) Nakayama lemma, this implies that

$$R^t \pi_{1,*}((\pi_2^* \dot{\omega}^\kappa)|_{(Z_T)_{\eta_T}}) = 0.$$

Indeed, write  $\mathcal{F}$  for  $(\pi_2^* \dot{\omega}^\kappa)|_{(Z_T)_{\eta_T}}$ . If  $\mathcal{I}$  is the ideal sheaf of  $(Z_T)_{\eta_T}^{\text{red}}$  in  $(Z_T)_{\eta_T}$ , it is enough to show that

$$R^t \pi_{1,*}(\mathcal{I}^i \mathcal{F} / \mathcal{I}^{i+1} \mathcal{F}) = 0 \quad \text{for every } i \geq 0.$$

But  $\mathcal{I}^i / \mathcal{I}^{i+1} \otimes \mathcal{F} \twoheadrightarrow \mathcal{I}^i \mathcal{F} / \mathcal{I}^{i+1} \mathcal{F}$ . By the right exactness of  $R^t \pi_{1,*}(-)$ , it suffices to show the vanishing of  $R^t \pi_{1,*}(\mathcal{I}^i / \mathcal{I}^{i+1} \otimes \mathcal{F})$ . But  $\mathcal{I}^i / \mathcal{I}^{i+1} \otimes \mathcal{F}$  is (scheme-theoretically) supported on  $(Z_T)_{\eta_T}^{\text{red}}$  and it receives generic surjective maps from finite direct sums of  $\mathcal{F}|_{(Z_T)_{\eta_T}^{\text{red}}}$  (for example induced by local generators of  $\mathcal{I}^i$ ). By the properties of  $R^t \pi_{1,*}(-)$  recalled above and the vanishing result (4.9.5), we deduce that  $R^t \pi_{1,*}(\mathcal{I}^i / \mathcal{I}^{i+1} \otimes \mathcal{F}) = 0$ . This concludes Proposition 3.18.  $\square$

## 5. RESULTS ON THE UNRAMIFIEDNESS OF MODULAR REPRESENTATIONS IN WEIGHT 1

Recall that  $\mathcal{O}$  denotes the ring of integers in a large enough finite extension  $E$  of  $\mathbb{Q}_p$ , with uniformizer  $\varpi$  and residue field  $\mathbb{F}$ . For simplicity, we assume for the entirety of this section that the prime  $p$  is *inert* in  $F$ , so that the Hecke operator  $T_p$  will be denoted by  $T_p$ . We denote by  $\epsilon : G_F \rightarrow \mathcal{O}^\times$  the  $p$ -adic cyclotomic character of  $G_F$ , and by  $\epsilon_m$  its reduction modulo  $\varpi^m$ .

Recall that  $\text{Sh}$  denotes the Hilbert modular Shimura scheme, smooth over  $\text{Spec } \mathcal{O}$ , of level  $\Gamma_{00}(\mathcal{N})$ . For any positive integer  $m$  denote by  $\text{Sh}_m$  the base change of  $\text{Sh} \rightarrow \text{Spec } \mathcal{O}$  to  $\text{Spec } (\mathcal{O}/(\varpi^m))$ , and similarly for  $\text{Sh}_m^{\text{tor}}, \omega_m^\kappa$ , etc. We denote simply by  $1$  the paritious parallel weight  $(1, 1) \in \mathbb{Z}^\Sigma \times \mathbb{Z}$ .

We assume throughout this section that  $p$  is odd.

**5.1. Hecke algebras.** We follow [CG12<sup>+</sup>] for most of the notation and constructions of this section. Let  $S$  denote a finite set of finite places of  $F$  containing the places above  $p\mathcal{N}$ , and let  $Q$  denote a finite set of finite places of  $F$  disjoint from  $S$ . (We will fix later suitable sets  $Q$  consisting of Taylor-Wiles primes; notice that  $Q = \emptyset$  is allowed for the moment). With abuse of notation, we will often use the letters  $S$  and  $Q$  also to denote the ideals of  $\mathcal{O}_F$  determined by the “product” of the places in  $S$  and  $Q$ , respectively.

Denote by  $\text{Sh}(Q)$  (resp.  $\text{Sh}(Q)_1$ ) the Shimura scheme over  $\text{Spec } \mathcal{O}$  of level  $\Gamma_{00}(\mathcal{N}) \cap \Gamma_0(Q)$  (resp. of level  $\Gamma_{00}(\mathcal{N}) \cap \Gamma_{00}(Q)$ ). Here for an ideal  $\mathfrak{a}$  of  $\mathcal{O}_F$  we set:

$$\Gamma_0(\mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_F) : c \in \mathfrak{a} \right\},$$

$$\Gamma_{00}(\mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{a}) : d - 1 \in \mathfrak{a} \right\}.$$

There is a natural étale morphism  $\text{Sh}(Q)_1 \rightarrow \text{Sh}(Q)$  with Galois group  $(\mathcal{O}_F/Q\mathcal{O}_F)^\times = \prod_{x \in Q} (\mathcal{O}_F/x)^\times$ . Let  $(\mathcal{O}_F/Q\mathcal{O}_F)^\times \twoheadrightarrow \Delta$  be a quotient map and denote by  $\text{Sh}(Q)_\Delta$  the corresponding subcover over  $\text{Sh}(Q)$  with Galois group  $\Delta$ . Similar constructions for the toroidal compactification and for the fiber at  $\mathcal{O}/(\varpi^m)$  give (with obvious notation) the covering map:

$$\text{Sh}(Q)_{\Delta,m}^{\text{tor}} \twoheadrightarrow \text{Sh}(Q)_m^{\text{tor}}.$$

Denote by  $\mathbb{T}_Q^{\text{univ}}$  the commutative polynomial algebra over the group-ring  $\mathcal{O}[(\mathcal{O}_F/\mathcal{N}Q\mathcal{O}_F)^\times]$  generated by the indeterminates  $t_x$  for finite places  $x \nmid p\mathcal{N}Q$  and  $u_x$  for finite places  $x \mid Q$ . The algebra  $\mathbb{T}_Q^{\text{univ}}$  acts on the coherent cohomology

$$(5.1.1) \quad \bigoplus_{\substack{k \geq 0, m \geq 1 \\ \kappa \text{ paritious}}} H^k(\text{Sh}(Q)_{\Delta, m}^{\text{tor}}, \omega_m^\kappa(-D))$$

via two quotients  $\mathbb{T}_Q^{\text{an}}$  and  $\mathbb{T}_Q$ : the first (resp. second) action is induced by the assignment  $t_x \mapsto T_x$  for  $x \nmid p\mathcal{N}Q$ , and  $u_x \mapsto 0$  (resp.  $u_x \mapsto U_x$ ) for  $x \mid Q$ , where  $T_x$  and  $U_x$  indicate the usual Hecke operators. In both cases, the group elements  $a \in (\mathcal{O}_F/\mathcal{N}Q\mathcal{O}_F)^\times$  act via the diamond operator  $\langle a \rangle$ . Recall that by construction the Hecke action preserves weights and degrees of cohomology. (Cf. [CG12<sup>+</sup>, Section 3.2.3, Definition 9.1]).

**Remark 5.2.** Properly speaking, we should denote the Hecke algebras just defined by  $\mathbb{T}_{Q, \Delta}^{\text{an}}$  and  $\mathbb{T}_{Q, \Delta}$ . By abuse of notation we drop the subscript  $\Delta$  and we allow the use of the same symbols to denote the action of the Hecke algebras with respect to different groups  $\Delta$ . Moreover, we warn the reader that later we will sometimes denote by  $\mathbb{T}_Q^{\text{an}}$  and  $\mathbb{T}_Q$  Hecke algebras acting on Hilbert modular classes of a fixed weight, or fixed cohomological degree; when doing so, we will explicitly state it.

**Remark 5.3.** We have an obvious injection  $\mathbb{T}_Q^{\text{an}} \hookrightarrow \mathbb{T}_Q$  and a natural map  $\mathbb{T}_Q^{\text{an}} \rightarrow \mathbb{T}_\emptyset^{\text{an}} = \mathbb{T}_\emptyset$  induced by the morphism  $\text{Sh}(Q)_\Delta \rightarrow \text{Sh}$ . Moreover the results of the previous section imply that the actions of  $\mathbb{T}_Q^{\text{an}}$  and  $\mathbb{T}_Q$  on coherent cohomology extend to an action of  $\mathbb{T}_Q^{\text{an}}[T_p]$  and  $\mathbb{T}_Q[T_p]$  respectively, as long as we consider in (5.1.1) only paritious weights  $\kappa = ((k_\tau)_\tau, w)$  such that  $k_\tau \geq 1$  for all  $\tau \in \Sigma$ . Following general conventions, we often denote by  $U_p$  the action of  $T_p$  on forms of weight  $\kappa$  when  $k_\tau \geq 2$  for all  $\tau \in \Sigma$  (cf. 3.13).

We fix a nebentypus character  $\varepsilon : (\mathcal{O}_F/\mathcal{N}Q\mathcal{O}_F)^\times \rightarrow \mathcal{O}^\times$  whose reduction modulo  $\varpi^m$  is also denoted  $\varepsilon$  when no confusion arises.

**5.4. Unramifiedness in cohomological degree zero.** In this section, for a set of places  $Q$  as above and a choice of  $\Delta$ , we let  $\mathbb{T}_Q$  denote the Hecke algebra generated by  $T_x$  ( $x \nmid p\mathcal{N}Q$ ),  $U_x$  ( $x \mid Q$ ), and the diamond operators acting on  $\bigoplus_{k \geq 0, m \geq 1} H^k(\text{Sh}(Q)_{\Delta, m}^{\text{tor}}, \omega_m^1(-D))$ . Moreover, we denote by  $\mathbb{T}_Q^{\{0\}}$  the quotient of  $\mathbb{T}_Q$  acting on  $\bigoplus_{m \geq 1} H^0(\text{Sh}(Q)_{\Delta, m}^{\text{tor}}, \omega_m^1(-D))$ .

Let  $\mathfrak{m}_\emptyset$  be a non-Eisenstein maximal ideal of  $\mathbb{T}_\emptyset^{\{0\}}$  with associated Galois representation  $\bar{\rho}$ . We fix a finite (possibly empty) set  $Q$  disjoint from  $S$ , consisting of finite places of  $F$  and satisfying:

- for each  $x \in Q$  we have  $x \equiv 1 \pmod{p}$ , and
- for each  $x \in Q$  the polynomial  $X^2 - T_x X + \langle x \rangle \in \mathbb{T}_\emptyset^{\{0\}}[X]$  has distinct roots modulo  $\mathfrak{m}_\emptyset$ ; we choose for each  $x \in Q$  one such root  $\alpha_x \in \mathbb{F}$ . (If necessary, we enlarge the field  $\mathbb{F}$ ).

We choose a surjection  $(\mathcal{O}_F/Q\mathcal{O}_F)^\times \twoheadrightarrow \Delta$  with  $\Delta$  a  $p$ -group.

Let  $\mathfrak{m}_Q$  denote the maximal ideal of  $\mathbb{T}_Q^{\{0\}}$  containing  $\mathfrak{m}_\emptyset$  and the elements  $U_x - \alpha_x$  for  $x \in Q$ . Moreover, denote by  $\mathfrak{m}'_Q$  the maximal ideal of  $\mathbb{T}_Q$  obtained by pulling back  $\mathfrak{m}_Q$  via the surjection  $\mathbb{T}_Q \twoheadrightarrow \mathbb{T}_Q^{\{0\}}$ . It follows from [ERX13<sup>+</sup>] that there is a Galois representation  $\rho'_Q : G_F \rightarrow \text{GL}_2(\mathbb{T}_{Q, \mathfrak{m}'_Q})$  lifting  $\bar{\rho}$ , unramified outside  $p\mathcal{N}Q$ , and such that  $\text{tr}(\rho'_Q(\text{Frob}_x)) = T_x$  for all  $x \nmid p\mathcal{N}Q$ . The following proposition shows that the representation obtained from  $\rho'_Q$  via the map  $\mathbb{T}_Q \twoheadrightarrow \mathbb{T}_Q^{\{0\}}$  is unramified at  $p$  (cf. [CG12<sup>+</sup>, Theorem 3.11], and also [DS74] and [Ed92, Proposition 2.7]).



**Proposition 5.5.** *Assume that for any lift  $\text{Frob}_p \in G_F$  of the arithmetic Frobenius at  $p$ , the eigenvalues of  $\bar{\rho}(\text{Frob}_p)$  in  $\bar{\mathbb{F}}$  are distinct. Then there exists a unique deformation*

$$\rho_Q : G_F \rightarrow \text{GL}_2(\mathbb{T}_{Q, \mathfrak{m}_Q}^{\{0\}})$$

*of  $\bar{\rho}$  unramified outside  $\mathcal{N}Q$  and such that for all primes  $x \nmid p\mathcal{N}Q$  we have  $\text{tr}(\rho_Q(\text{Frob}_x)) = T_x$ . In particular,  $\rho_Q$  is unramified at  $p$ .*

*Proof.* Thanks to the existence and properties of the operator  $T_p$  acting on weight one forms (cf. section 3.9), we can prove the result *exactly* as in [CG12<sup>+</sup>, Theorem 3.11], where the case  $F = \mathbb{Q}$  is treated. For completeness, we sketch the argument below, but the reader is referred to *loc.cit.* for further details.

We set  $\mathbf{p} := (p, \dots, p)$ ,  $\mathbf{1} := (1, \dots, 1) \in \mathbb{Z}^\Sigma$ . Let  $M$  be a positive integer divisible by  $p^{m-1}$  and denote by  $\tilde{h}_M \in H^0(\text{Sh}(Q)_{\Delta, m}^{\text{tor}}, \omega_m^{(M(\mathbf{p}-\mathbf{1}), M(p-1))})$  a lift to  $\mathcal{O}/(\varpi^m)$  of the  $M$ th power of the total Hasse invariant  $h \in H^0(\text{Sh}(Q)_{\Delta, \mathbb{F}}^{\text{tor}}, \omega_{\mathbb{F}}^{(\mathbf{p}-\mathbf{1}, p-1)})$  (cf. [ERX13<sup>+</sup>, 3.3.1]). Let  $U_p$  denote the action of the Hecke operator  $T_p$  on modular forms of paritious weight  $(\mathbf{n}, n) := (\mathbf{1} + M(\mathbf{p} - \mathbf{1}), 1 + M(p - 1))$  over  $\mathcal{O}/(\varpi^m)$ , so that  $U_p$  acts on  $q$ -expansions by  $\sum_{\alpha} a_{\alpha} q^{\alpha} \mapsto \sum_{\alpha} a_{p\alpha} q^{\alpha}$  by Remark 3.13. Define the operator

$$V_M := \varepsilon(p)^{-1} \cdot (\tilde{h}_M \circ T_p - U_p \circ \tilde{h}_M)^{17}$$

sending  $H^0(\text{Sh}(Q)_{\Delta, m}^{\text{tor}}, \omega_m^1(-D))$  into  $H^0(\text{Sh}(Q)_{\Delta, m}^{\text{tor}}, \omega_m^{(\mathbf{n}, n)}(-D))$ . Notice that  $V_M$  is well defined, since the hypothesis on our weights guarantees that  $T_p$  and  $U_p$  are defined. Using the fact that the  $q$ -expansion of  $\tilde{h}_M$  is one at each cusp of  $\text{Sh}_m^{\text{tor}}$ , together with Remark 3.13 applied in weight 1, we see that the action of  $V_M$  on  $q$ -expansion is given by  $\sum_{\alpha} a_{\alpha} q^{\alpha} \mapsto \sum_{\alpha} a_{\alpha} q^{p\alpha}$ .

We claim that the forms  $\tilde{h}_M f$  and  $V_M f$  in  $H^0(\text{Sh}(Q)_{\Delta, m}^{\text{tor}}, \omega_m^{(\mathbf{n}, n)}(-D))$  are  $\mathcal{O}/(\varpi^m)$ -linearly independent. It is enough to work modulo  $\varpi$ ; denote by  $\bar{f}$  the reduction modulo  $\varpi$  of  $f$ . Clearly  $h^M \bar{f}$  and  $V_M \bar{f}$  are non-zero since both  $h^M$  and  $V_M$  are injective on forms modulo  $\varpi$ . So assume that

$$(5.5.1) \quad h^M \bar{f} = a \cdot V_M \bar{f}$$

in  $\mathbb{F}[[q^{\alpha}]]$  for some  $a \in \mathbb{F}^{\times}$ . Recall that there is a differential operator  $\theta$  acting on Hilbert modular forms over  $\mathbb{F}$  and increasing weight by  $(p+1, \dots, p+1)$ , whose action on  $q$ -expansion mod  $p$  is given by:  $\sum_{\alpha} a_{\alpha} q^{\alpha} \mapsto \sum_{\alpha} \overline{\text{Nm}_{F/\mathbb{Q}}(\alpha)} \cdot a_{\alpha} q^{\alpha}$  (cf. [AG05, 16.2]). Applying  $\theta$  to both sides of (5.5.1), we obtain that  $\theta(h^M \bar{f}) = 0$ . Since  $\theta$  and  $h$  commute (we can check this on  $q$ -expansion), the injectivity of  $h$  implies that  $\bar{f} \in \ker \theta$ . We conclude that  $\bar{f} = 0$ , since  $\theta$  has trivial kernel in weight  $(\mathbf{1}, 1)$ . (This last fact follows from the arguments of [Ka76, IV], suitably extended to the settings of Hilbert modular forms).

We conclude that the map  $(\tilde{h}^M, \varepsilon(p)V_M)$  induces an embedding

$$\psi : H^0(\text{Sh}(Q)_{\Delta, m}^{\text{tor}}, \omega_m^1(-D))^{\oplus 2} \hookrightarrow H^0(\text{Sh}(Q)_{\Delta, m}^{\text{tor}}, \omega_m^{(\mathbf{n}, n)}(-D))$$

which is equivariant under the action of  $\mathbb{T}_Q^{\{0\}}$ . The action of  $U_p$  on the domain of  $\psi$  is then given, via  $\psi^{-1}$ , by the matrix

$$\begin{pmatrix} T_p & 1 \\ -\langle p \rangle & 0 \end{pmatrix},$$

and  $U_p$  satisfies  $X^2 - T_p X + \langle p \rangle = 0$ . Denote by  $\alpha$  and  $\beta$  the distinct eigenvalues of  $\bar{\rho}(\text{Frob}_p)$  in  $\bar{\mathbb{F}}$ , and choose lifts  $\tilde{\alpha}$  and  $\tilde{\beta}$  of  $\alpha$  and  $\beta$  respectively to  $\mathcal{O}$  (for this we might need to enlarge

<sup>17</sup>If  $m = 1$  we can choose  $M = 1$  and then  $V_M$  coincides with the classical operator  $V_p$  induced by Frobenius base-change on the abelian schemes parametrized by  $\text{Sh} \otimes \mathbb{F}$ .

$E$ ). We have  $\alpha\beta \equiv \langle p \rangle \pmod{\mathfrak{m}_Q}$ , and the Hecke operator  $(U_p - \tilde{\alpha})(U_p - \tilde{\beta})$  acts nilpotently on  $\psi(H^0(\mathrm{Sh}(Q)_{\Delta,m}^{\mathrm{tor}}, \omega_m^1(-\mathbb{D}))_{\mathfrak{m}_Q}^{\oplus 2})$ .

We denote by  $\mathbb{T}_{Q,n}^{\{0\}}$  the Hecke algebra acting on  $\oplus_{m \geq 1} H^0(\mathrm{Sh}(Q)_{\Delta,m}^{\mathrm{tor}}, \omega_m^{(\mathbf{n},n)}(-\mathbb{D}))$  generated by the operators  $T_x$  for  $x \nmid p\mathcal{N}Q$ ,  $U_x$  for  $x \mid Q$ , and the diamond operators. We set  $\tilde{\mathbb{T}}_{Q,n}^{\{0\}} := \mathbb{T}_{Q,n}^{\{0\}}[U_p]$ . By abuse of notation, denote by  $\mathfrak{m}_Q$  the maximal ideal of  $\mathbb{T}_{Q,n}^{\{0\}}$  containing  $\mathfrak{m}_\emptyset$  and  $U_x - \alpha_x$  for all  $x \in Q$ . Let moreover  $\tilde{\mathfrak{m}}_\alpha$  (resp.  $\tilde{\mathfrak{m}}_\beta$ ) denote the maximal ideal of  $\tilde{\mathbb{T}}_{Q,n}^{\{0\}}$  containing  $\mathfrak{m}_Q$  and  $U_p - \alpha$  (resp.  $U_p - \beta$ ).

Let  $I_m$  denote the annihilator in  $\mathbb{T}_{Q,\mathfrak{m}_Q}^{\{0\}}$  of  $H^0(\mathrm{Sh}(Q)_{\Delta,m}^{\mathrm{tor}}, \omega_m^1(-\mathbb{D}))_{\mathfrak{m}_Q}$ . There is a natural map  $\mathbb{T}_{Q,n,\mathfrak{m}_Q}^{\{0\}} \rightarrow \tilde{\mathbb{T}}_{Q,n,\tilde{\mathfrak{m}}_\alpha}^{\{0\}}$  whose image we denote  $\mathbb{T}_{Q,n,\mathfrak{m}_\alpha}^{\{0\}}$ . As in [CG12<sup>+</sup>, 3.5], we see that  $\mathbb{T}_{Q,\mathfrak{m}_Q}^{\{0\}}/I_m[U_p] \subset \mathrm{End}_{\mathcal{O}}(\mathrm{Im}\psi)_{\tilde{\mathfrak{m}}_\alpha}$  contains  $T_p$  and is naturally a quotient of  $\tilde{\mathbb{T}}_{Q,n,\tilde{\mathfrak{m}}_\alpha}^{\{0\}}$ ; correspondingly,  $\mathbb{T}_{Q,\mathfrak{m}_Q}^{\{0\}}/I_m$  is a quotient of  $\mathbb{T}_{Q,n,\mathfrak{m}_\alpha}^{\{0\}}$ . Denote by  $\rho_{Q,n,\alpha} : G_F \rightarrow \mathrm{GL}_2(\mathbb{T}_{Q,n,\mathfrak{m}_\alpha}^{\{0\}})$  the Galois representation attached to the ordinary Hecke algebra acting in weight  $(\mathbf{n}, n)$  and cohomological degree zero, and similarly for  $\tilde{\rho}_{Q,n,\alpha} : G_F \rightarrow \mathrm{GL}_2(\tilde{\mathbb{T}}_{Q,n,\tilde{\mathfrak{m}}_\alpha}^{\{0\}})$ . Composing these representations with the two quotient maps considered above, we obtain representations  $\rho_{Q,m}$  and  $\tilde{\rho}_{Q,m}$ . The representation  $\rho_Q := \varprojlim_m \rho_{Q,m}$  satisfies the desired properties, except possibly the condition of being unramified at  $p$ . We observe that

$$\tilde{\rho}_{Q,m}|_{G_{F_p}} \simeq \begin{pmatrix} \epsilon_m^{M(p-1)} \lambda_{\tilde{\beta}} & * \\ 0 & \lambda_{\tilde{\alpha}} \end{pmatrix}$$

where  $\lambda_x : G_{F_p} \rightarrow (\mathcal{O}/(\varpi^m))^\times$  denotes the unramified character of  $G_{F_p}$  sending a geometric Frobenius element to  $x$ . Notice that  $\epsilon_m^{M(p-1)}$  is trivial since  $p^{m-1}$  divides  $M$ .

The Galois representation  $\tilde{\rho}_{Q,m}$  can be equivalently (by the Chebotarev density theorem) defined using the eigenvalue  $\beta$ , so that:

$$\tilde{\rho}_{Q,m}|_{G_{F_p}} \simeq \begin{pmatrix} \lambda_{\tilde{\alpha}} & * \\ 0 & \lambda_{\tilde{\beta}} \end{pmatrix} \simeq \begin{pmatrix} \lambda_{\tilde{\beta}} & * \\ 0 & \lambda_{\tilde{\alpha}} \end{pmatrix},$$

Since  $\tilde{\alpha} \neq \tilde{\beta}$  we deduce that the extension classes denoted by  $*$  are trivial, and the result follows.  $\square$

**Remark 5.6.** It seems that the methods of [CG12<sup>+</sup>, 3.6-7] would allow us to prove the above result also when  $\alpha = \beta$ .

**5.7. Unramifiedness in the case of surfaces.** We assume in this section that  $g = 2$ . Recall that we are moreover requiring for simplicity that  $p$  is inert in  $F$ . We will prove, under the assumption of Frobenius-distinguishness introduced in Proposition 5.5, that Galois representations arising from Hilbert modular classes of paritious weights  $\kappa = (\mathbf{1}, 1)$  are unramified at  $p$ .

**Remark 5.8.** As mentioned earlier, the assumption of Frobenius distinguishness seems unnecessary. Moreover, we can prove the above mentioned result for general totally real fields  $F$  and general  $p$ , as long as for each  $\mathfrak{p}|p$  we have  $f_{\mathfrak{p}}e_{\mathfrak{p}} \leq 2$ . For simplicity, we treat here only the case of surfaces with  $p$  inert, and we will work in more general settings in a later version of this paper.

Let  $\tau_m : G_{F,S} \rightarrow \mathcal{O}/(\varpi^m)$  be a continuous, two-dimensional,  $\mathcal{O}/(\varpi^m)$ -linear Galois pseudo-representation. Suppose that  $\tau_m$  is *modular of weight 1*, in the sense that there is an integer  $j(\tau_m)$  and a cohomology class  $c \in H^{j(\tau_m)}(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-\mathbb{D}))$  of nebentypus  $\varepsilon$  such that  $T_{\mathfrak{q}}c = a_{\mathfrak{q}}c$  and  $\tau_m(\mathrm{Frob}_{\mathfrak{q}}) = a_{\mathfrak{q}}$  for all finite places  $\mathfrak{q}$  of  $F$  outside  $S$  (here  $a_{\mathfrak{q}} \in \mathcal{O}/(\varpi^m)$ ). Denote by  $\bar{\rho}$  the

semisimple Galois representation associated to the reduction modulo  $\varpi$  of  $\tau_m$ , and let  $\mathfrak{m}_\emptyset$  denote the maximal ideal of  $\mathbb{T}_\emptyset$  associated to  $\bar{\rho}$ , *i.e.*:

$$\mathfrak{m}_\emptyset = (\varpi, T_q - \text{tr} \bar{\rho}(\text{Frob}_q) : q \notin S; \langle q \rangle - \det \bar{\rho}(\text{Frob}_q) : q \nmid \mathcal{N}).$$

**Assumption.** We assume that  $\bar{\rho}$  is absolutely irreducible, and that the eigenvalues (in  $\overline{\mathbb{F}}$ ) of any lift of a Frobenius element at  $p$  are distinct (Frobenius-distinguishness).

Let  $\chi$  denote the Teichmüller lift of  $\det \bar{\rho}$  and denote by  $R_p$  the complete local Noetherian  $\mathcal{O}$ -algebra representing the functor of framed  $\mathcal{O}$ -deformations of  $\bar{\rho}|_{G_{F_p}}$  with determinant  $\chi|_{G_{F_p}}$ .

Denote by  $\mathbb{T}_{\emptyset, \mathfrak{m}_\emptyset}$  the completion at  $\mathfrak{m}_\emptyset$  of the tame Hecke algebra  $\mathbb{T}_\emptyset$  acting by  $\mathcal{O}$ -linear endomorphisms on the space (5.1.1) when  $Q = \emptyset$ . The main result of [ERX13<sup>+</sup>] implies that there is a natural continuous homomorphism of  $\mathcal{O}$ -algebras  $R_p \rightarrow \mathbb{T}_{\emptyset, \mathfrak{m}_\emptyset}$ . In particular, we can view the completions at  $\mathfrak{m}_\emptyset$  of the coherent cohomology of  $\text{Sh}_m^{\text{tor}}$  and  $\text{Sh}^{\text{tor}}$  as  $R_p$ -modules.

The exact sequence:

$$0 \rightarrow \omega^1 \xrightarrow{\cdot \varpi^m} \omega^1 \rightarrow \omega_m^1 \rightarrow 0$$

of coherent sheaves on  $\text{Sh}_\emptyset^{\text{tor}}$  induces a long exact sequence in cohomology which, after localization at  $\mathfrak{m}_\emptyset$ , is given by:

$$(5.8.1) \quad \cdots \rightarrow H^i(\text{Sh}^{\text{tor}}, \omega^1(-D))_{\mathfrak{m}_\emptyset} \xrightarrow{\cdot \varpi^m} H^i(\text{Sh}^{\text{tor}}, \omega^1(-D))_{\mathfrak{m}_\emptyset} \rightarrow H^i(\text{Sh}_m^{\text{tor}}, \omega_m^1(-D))_{\mathfrak{m}_\emptyset} \rightarrow H^{i+1}(\text{Sh}^{\text{tor}}, \omega^1(-D))_{\mathfrak{m}_\emptyset} \rightarrow \cdots$$

**Remark 5.9.** Let  $f : \text{Sh}_B^{\text{tor}} \rightarrow \text{Spec } B$  be the smooth structure morphism of  $\text{Sh}_B^{\text{tor}}$ , where  $B$  is either a non-zero quotient of  $\mathcal{O}$ , or its fraction field  $E$ . If  $M$  is an injective  $B$ -module and  $\mathcal{L}$  is an invertible sheaf on  $\text{Sh}_B^{\text{tor}}$ , we have by Grothendieck-Serre-Verdier duality a canonical isomorphism (cf. [Ha66, III.11, Corollary 11.2]):

$$D : \text{Hom}_B(H^i(\text{Sh}_B^{\text{tor}}, \mathcal{L}^{\otimes -1} \otimes \mathcal{D} \otimes f^* M), M) \xrightarrow{\sim} H^{g-i}(\text{Sh}_B^{\text{tor}}, \mathcal{L}),$$

where  $\mathcal{D}$  denotes the dualizing sheaf of  $f$ . When  $\mathcal{L}$  is an automorphic line bundle, the behavior of this isomorphism with respect to the Hecke action is described in [CG12<sup>+</sup>, page 19-20]:

$$D(T_x c) = \langle x \rangle^{-1} T_x D(c),$$

$$D(\langle a \rangle c) = \langle a \rangle^{-1} D(c),$$

where  $c$  denotes a cohomology class,  $x$  is a place of  $F$  such that  $x \nmid p\mathcal{N}$ , and  $a \in (\mathcal{O}_F/\mathcal{N}\mathcal{O}_F)^\times$ .

Denote by  $\mathcal{I}$  the ideal of  $R_p$  characterized by the following property: a lifting  $\rho : G_{F,S} \rightarrow GL_2(A)$  of  $\bar{\rho}$  with values in a complete local noetherian  $\mathcal{O}$ -algebra  $A$  is unramified at the unique prime of  $F$  above  $p$  if and only if the corresponding map  $R_p \rightarrow A$  factors via  $R_p/\mathcal{I}$ .

**Lemma 5.10.** *If  $i \in \{0, 2\}$ , the  $R_p$ -module  $H^i(\text{Sh}_m^{\text{tor}}, \omega_m^1(-D))_{\mathfrak{m}_\emptyset}$  is supported on  $\text{Spec}(R_p/\mathcal{I})$ . In particular, if  $j(\tau_m) \in \{0, 2\}$  then  $\bar{\rho}$  is unramified at  $p$ .*

*Proof.* When  $i = 0$  this follows from the Frobenius distinguishness assumption and Proposition 5.5. Assume  $i = 2$ , and recall that the Kodaira-Spencer morphism induces an identification

$$\Omega_{\text{Sh}_m^{\text{tor}}/\mathcal{O}/(\varpi^m)}^2 \simeq \omega_m^{(2,0)}(-D).$$

Since  $\mathcal{O}/(\varpi^m)$  is an injective  $\mathcal{O}/(\varpi^m)$ -module, Grothendieck-Serre-Verdier duality gives an isomorphism:

$$\begin{aligned} H^2(\text{Sh}_m^{\text{tor}}, \omega_m^1(-D))_{\mathfrak{m}_\emptyset} &= H^2(\text{Sh}_m^{\text{tor}}, (\omega_m^{(1,-1)})^{\otimes -1} \otimes \omega_m^{(2,0)}(-D))_{\mathfrak{m}_\emptyset} \\ &\simeq H^0(\text{Sh}_m^{\text{tor}}, \omega_m^{(1,-1)})_{\mathfrak{m}_\emptyset}^\vee, \end{aligned}$$

where  $()^\vee$  denotes  $\mathcal{O}/(\varpi^m)$ -dual. The space  $H^0(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^{(1,-1)})_{\mathfrak{m}_0}^\vee$  gives rise to representations unramified at  $p$ : this follows again from the proof of Proposition 5.5.  $\square$

**Lemma 5.11.** *The representation  $\bar{\rho}$  is unramified at  $p$ .*

*Proof.* Suppose by contradiction that  $\bar{\rho}$  is ramified at  $p$ . Since  $H^0(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D)) \subset \prod_{\mathfrak{c}} \mathcal{O}[[q^\alpha]]$  is  $\varpi$ -torsion free, it is well known that the representations arising from  $\mathbb{T}_\emptyset$ -eigenforms in this space are unramified at  $p$ . In particular  $H^0(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}_0} = 0$ . By Lemma 5.10 we also see that  $H^i(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}_0} = 0$  for  $i \in \{0, 2\}$  and for any  $m$ .

The exact sequence (5.8.1) gives:

$$\begin{aligned} 0 \rightarrow H^1(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D))_{\mathfrak{m}_0} &\xrightarrow{\varpi^m} H^1(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}_0} \rightarrow H^1(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}_0} \\ &\rightarrow H^2(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D))_{\mathfrak{m}_0} \xrightarrow{\varpi^m} H^2(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}_0} \rightarrow 0. \end{aligned}$$

The injectivity of the second map implies that  $H^1(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D))_{\mathfrak{m}_0}$  is  $\varpi$ -torsion free, and hence zero since Galois representations arising from  $H^1(\mathrm{Sh}_{\mathbb{Q}_p}^{\mathrm{tor}}, \omega_{\mathbb{Q}_p}^1(-D))$  are unramified at  $p$ . The surjectivity of the multiplication-by- $\varpi^m$  map between the degree-two cohomology groups implies that those localized modules are zero. We conclude that  $H^1(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}_0} = 0$ , and hence the ideal  $\mathfrak{m}_0$  is not in the support of any cohomology, contradicting the modularity of  $\bar{\rho}$ .  $\square$

Notice that  $\mathcal{I}$  is a proper ideal of  $R_p$ , since  $\bar{\rho}$  is unramified by Lemma 5.11.

**Proposition 5.12.** *There is a positive integer  $n$  such that the  $R_p$ -modules  $H^i(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D))_{\mathfrak{m}_0}$  and  $H^i(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}_0}$  are annihilated by  $\mathcal{I}^n$  for all  $i, m$ . Moreover, the value  $n = 4$  suffices (recall that we are assuming  $g = 2$ )<sup>18</sup>.*

*Proof.* We argue using the long exact sequence of  $R_p$ -modules (5.8.1). First of all  $H^0(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D))_{\mathfrak{m}_0}$  is  $\varpi$ -torsion free, hence a Hecke-equivariant subspace of  $H^0(\mathrm{Sh}_E^{\mathrm{tor}}, \omega_E^1(-D))_{\mathfrak{m}_0}$ ; the classical complex theory implies therefore that it is annihilated by  $\mathcal{I}$ . Moreover by Lemma 5.10 we know that also  $H^0(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}_0}$  and  $H^2(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}_0}$  are annihilated by  $\mathcal{I}$ .

We now look at the exact sequence of  $R_p$ -modules:

$$(5.12.1) \quad 0 \rightarrow H^1(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D))_{\mathfrak{m}_0}[\varpi^\infty] \rightarrow H^1(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D))_{\mathfrak{m}_0} \rightarrow E \otimes_{\mathcal{O}} H^1(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D))_{\mathfrak{m}_0} \rightarrow 0.$$

The second term in the above sequence is a quotient of  $H^0(\mathrm{Sh}_{m'}^{\mathrm{tor}}, \omega_{m'}^1(-D))_{\mathfrak{m}_0}$  for some large enough  $m' > 0$ ; in particular as an  $R_p$ -module it is annihilated by  $\mathcal{I}$ . The fourth term of (5.12.1) coincides with  $H^1(\mathrm{Sh}_E^{\mathrm{tor}}, \omega_E^1(-D))_{\mathfrak{m}_0}$  and is therefore annihilated by  $\mathcal{I}$ . We conclude that  $\mathcal{I}^2 \cdot H^1(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D))_{\mathfrak{m}_0} = 0$ .

The short exact sequence of coherent sheaves on  $\mathrm{Sh}$ :

$$0 \rightarrow \omega_m^1 \rightarrow E/\mathcal{O} \otimes_{\mathcal{O}} \omega^1 \xrightarrow{\varpi^m} E/\mathcal{O} \otimes_{\mathcal{O}} \omega^1 \rightarrow 0$$

induces the following exact sequence of  $R_p$ -modules:

$$(5.12.2) \quad H^0(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D) \otimes E/\mathcal{O})_{\mathfrak{m}_0} \rightarrow H^1(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}_0} \rightarrow H^1(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D) \otimes E/\mathcal{O})_{\mathfrak{m}_0}.$$

The first term equals  $\varinjlim_{m'} H^0(\mathrm{Sh}_{m'}^{\mathrm{tor}}, \omega_{m'}^1(-D))_{\mathfrak{m}_0}$ , and is therefore annihilated by  $\mathcal{I}$ . Grothendieck-Serre-Verdier duality gives:

$$\begin{aligned} H^1(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D) \otimes E/\mathcal{O})_{\mathfrak{m}_0} &= H^1(\mathrm{Sh}^{\mathrm{tor}}, (\omega^{(1,-1)})^{\otimes -1} \otimes \Omega_{\mathrm{Sh}^{\mathrm{tor}}}^2 \otimes E/\mathcal{O})_{\mathfrak{m}_0} \\ &\simeq H^1(\mathrm{Sh}^{\mathrm{tor}}, \omega^{(1,-1)})_{\mathfrak{m}_0}^\vee, \end{aligned}$$

<sup>18</sup>It is possible that the value  $n = 2$  would suffice, when  $g = 2$ .

where  $(\cdot)^\vee$  indicates Pontryagin dual. In particular, our previous arguments imply that  $H^1(\mathrm{Sh}^{\mathrm{tor}}, \omega^{(1,-1)})_{\mathfrak{m}_\emptyset}^\vee$  is annihilated by  $\mathcal{I}^2$ , and so the same holds for  $H^1(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D) \otimes E/\mathcal{O})_{\mathfrak{m}_\emptyset}$ . Therefore  $H^1(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}_\emptyset}$  is annihilated by  $\mathcal{I}^3$  by (5.12.2).

To conclude, using the analogue of (5.12.1) for degree two cohomology and the fact that  $\mathcal{I}^3 \cdot H^1(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}_\emptyset} = 0$ , we obtain that  $H^2(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D))_{\mathfrak{m}_\emptyset}$  is annihilated by  $\mathcal{I}^4$ .  $\square$

**5.13. The patching argument.** We make use here the techniques of patching complexes of [CG12<sup>+</sup>, 6]. All the ring dimensions computed below are absolute Krull dimensions, unless otherwise stated. Recall that  $p$  is assumed to be inert in the totally real field  $F$ , and that we assume  $g = 2$  (but most of the arguments below continue to hold for arbitrary  $g$ ).

Let  $\mathcal{R}$  denote the set of places of  $F$  dividing the level  $\mathcal{N}$  of the Shimura variety  $\mathrm{Sh}$ , and let  $S = \{p\} \cup \mathcal{R}$ . As before, we fix a finite set  $Q$  of finite places of  $F$  disjoint from  $S$ .

Recall that  $R_p$  denotes the universal *framed*  $\mathcal{O}$ -deformation ring of  $\bar{\rho}|_{G_{F_p}}$  corresponding to lifts of determinant  $\chi|_{G_p}$ . For any positive integer  $n$  we let  $R_p^{(n)} := R_p/\mathcal{I}^n$ . Since unramified lifts are determined by the matrix of a Frobenius element, we deduce by simple calculations that:

$$\dim R_p^{(n)} = 4.$$

For each  $x \in \mathcal{R}$  we similarly denote by  $R_x$  the universal framed  $\mathcal{O}$ -deformation ring of  $\bar{\rho}|_{G_{F_x}}$  corresponding to lifts of determinant  $\chi|_{G_x}$ . Since  $x \nmid p$  we have (cf. [Boe11, Theorem 3.3.1]):

$$\dim R_x = 4, \quad x \in \mathcal{R}.$$

Let

$$R_{\mathrm{loc}} := R_p^{(n)} \hat{\otimes}_{\mathcal{O}} \bigotimes_{x \in \mathcal{R}} R_x.$$

We denote by  $R_Q^{(n)}$  (resp.  $R_Q^{(n), \square_S}$ ) the universal  $\mathcal{O}$ -deformation ring of  $\bar{\rho}$  corresponding to those deformations which are:

- unramified outside  $S \cup Q$ ;
- of determinant  $\chi$ ;
- of the type classified by  $R_p^{(n)}$  at  $p$ ;
- (resp. and framed at the places in  $S$ ).

Restriction to decomposition groups at the places in  $S$  induces a natural morphism  $R_{\mathrm{loc}} \rightarrow R_Q^{(n), \square_S}$ . Moreover the obvious forgetful map  $R_Q^{(n)} \rightarrow R_Q^{(n), \square_S}$  is formally smooth of relative Krull dimension (cf. [Boe11, Corollary 5.2.1]):

$$j := 4|S| - 1.$$

Assume that  $\bar{\rho}(G_F)$  contains  $\mathrm{SL}_2(\mathbb{F}_p)$  and that  $p > 3$ . Recall that  $\bar{\rho}$  is totally odd, since it is modular. By [Ge14, Proposition 5.9]<sup>19</sup> there exists an integer  $q \geq 1$  with the following property: for any  $N \geq 1$  there is a set  $Q_N$  consisting of finite primes of  $F$  such that:

- $Q_N$  has cardinality  $q$  and is disjoint from the set  $S$ ;
- for each  $x \in Q_N$ ,  $\bar{\rho}(\mathrm{Frob}_x)$  has two distinct eigenvalues  $\alpha_x, \beta_x \in \mathbb{F}$ ;
- $\mathrm{Nm}_{F/\mathbb{Q}}(x) \equiv 1 \pmod{p^N}$  for all  $x \in Q_N$ ;
- $R_{Q_N}^{(n), \square_S}$  is topologically generated over  $R_{\mathrm{loc}}$  by  $h := q + |S| - 1 - [F : \mathbb{Q}] = q + |S| - 3$  elements.

<sup>19</sup>Notice that what we denote  $S$  is denoted by  $T$  in *loc.cit.*.

For each  $N$  we now fix a choice of Taylor-Wiles primes  $Q_N$ , and for each such set a choice of distinguished eigenvalue  $\alpha_x$  of  $\bar{\rho}(\text{Frob}_x)$  for  $x \mid Q_N$ .

Set  $n = 4$  and define

$$R_\infty := R_{\text{loc}}[[x_1, \dots, x_h]],$$

so that

$$(5.13.1) \quad \dim R_\infty = 3|S| + 1 + h = 1 + q + j - [F : \mathbb{Q}].$$

We choose for each  $N \geq 1$  a surjection

$$(5.13.2) \quad R_\infty \twoheadrightarrow R_{Q_N}^{(4), \square_S}.$$

We let  $S_N = \mathcal{O}[\Delta_N]$ , where  $\Delta_N = (\mathbb{Z}/p^N\mathbb{Z})^q$ , and we set  $S_\infty = \varprojlim_N S_N \simeq \mathcal{O}[(\mathbb{Z}_p)^q]$ . If  $M \geq N \geq 0$  and if  $I$  is an ideal of  $\mathcal{O}$ , we regard  $S_N/I$  as a quotient of  $S_M$  via the natural surjective maps  $\mathcal{O} \rightarrow \mathcal{O}/I$  and  $\Delta_M \rightarrow \Delta_N$ . We denote the operation of complete tensor product over  $\mathcal{O}$  with  $\mathcal{O}^\square := \mathcal{O}[[z_1, \dots, z_j]]$  by the superscript  $\square$ .

Denote by  $\mathfrak{m}_{Q_N}$  the maximal ideal of the Hecke algebra  $\mathbb{T}_{Q_N}$  contracting to  $\mathfrak{m}_\emptyset \subset \mathbb{T}_\emptyset$  and containing  $U_x - \alpha_x$  for each  $x \mid Q_N$ , where the eigenvalues  $\alpha_x$  are fixed as above. Applying to our settings the construction of [CG12<sup>+</sup>, 7.2], we deduce the existence of a perfect complex

$$0 \rightarrow C_{N,2} \rightarrow C_{N,1} \rightarrow C_{N,0} \rightarrow 0$$

of  $S_N/(\varpi^N)$ -modules with an action of  $\mathbb{T}_{Q_N, \mathfrak{m}_{Q_N}}$  whose  $i$ th homology is  $\mathbb{T}_{Q_N, \mathfrak{m}_{Q_N}}$ -equivariantly isomorphic to:

$$\begin{aligned} H_i(\text{Sh}(Q_N)_{\Delta_N, N}^{\text{tor}}, \omega_N^{(1, -1)})_{\mathfrak{m}_{Q_N}} &:= H^i(\text{Sh}(Q_N)_{\Delta_N, N}^{\text{tor}}, (\omega_N^{(1, -1)})^{\otimes -1} \otimes \Omega^2)_{\mathfrak{m}_{Q_N}}^\vee \\ &\simeq H^i(\text{Sh}(Q_N)_{\Delta_N, N}^{\text{tor}}, \omega_N^1(-\mathbb{D}))_{\mathfrak{m}_{Q_N}}^\vee. \end{aligned}$$

Here  $\Omega^2$  denotes the canonical sheaf of the smooth  $\mathcal{O}/(\varpi^N)$ -scheme  $\text{Sh}(Q_N)_{\Delta_N, N}^{\text{tor}}$ , and the superscript  $^\vee$  denotes taking  $\mathcal{O}/(\varpi^N)$ -dual.

**Remark 5.14.** The complex that we have denoted here by  $C_{N,*}$  is denoted in [CG12<sup>+</sup>, page 74] by

$$\varinjlim_m T^m C_*$$

where  $T$  is a suitable Hecke operator constructed at the end of paragraph 7.1 of *op.cit.* Taking the limit is what allows to obtain the cohomology localized at  $\mathfrak{m}_{Q_N}$  in [CG12<sup>+</sup>].

Let  $D_N^*$  denote the chain complex obtained from  $C_{N,*}$  by taking  $\mathcal{O}/(\varpi^N)$ -duals, i.e.,  $D_N^i := C_{N,i}^\vee$ . Observe that

$$(5.14.1) \quad H^i(D_N^*) \simeq H^i(\text{Sh}(Q_N)_{\Delta_N, N}^{\text{tor}}, \omega_N^1(-\mathbb{D}))_{\mathfrak{m}_{Q_N}}$$

is the cohomology in which we are interested. The main result of [ERX13<sup>+</sup>] together with Proposition 5.12 imply the existence of a canonical map  $R_{Q_N}^{(4), \square_S} \rightarrow \mathbb{T}_{Q_N, \mathfrak{m}_{Q_N}}^\square$ , so that the global deformation ring  $R_{Q_N}^{(4), \square_S}$  acts on the cohomology of  $D_N^*$ . In particular for each  $M \geq N \geq 0$  with  $M \geq 1$  and for each  $m \geq 1$ , the cohomology  $H^i(D_M^{\square,*} \otimes_{S_M} S_N/(\varpi^m))$  is also an  $R_\infty$ -module via (5.13.2), and the actions of  $R_\infty$  and  $S_M^\square$  on this space commute.

We let  $H := H^2(\text{Sh}^{\text{tor}}, \omega^1(-\mathbb{D}) \otimes_{\mathcal{O}} E/\mathcal{O})_{\mathfrak{m}_\emptyset}$  and we defined a chain complex  $T$  with trivial differentials  $d = 0$  by setting  $T^i := H^i(\text{Sh}_{\mathbb{F}}^{\text{tor}}, \omega_{\mathbb{F}}^1(-\mathbb{D}))_{\mathfrak{m}_\emptyset}$ , so that  $H^2(T) \simeq H/(\varpi)$ . We also set  $R = R_\emptyset^{(4), \square_S}$  to be the global deformation ring defined earlier attached to the empty set of Taylor-Wiles primes. Notice that  $H$  is an  $R$ -module.

By (5.13.1) we have the numerical equality:

$$\dim R_\infty = \dim S_\infty^\square - [F : \mathbb{Q}].$$

We then see that the hypotheses of Theorem 6.3 of [CG12<sup>+</sup>] are satisfied (the notation we introduced for  $S_N, S_\infty, R, R_\infty, H, T$ , and  $D_N$  matches the notation of the statement of the theorem in *loc.cit.*, with  $l_0$  there being equal to  $g = 2$ ). In particular we can patch the complexes  $D_N^*$  to produce a perfect chain complex

$$0 \rightarrow P_\infty^{\square,0} \rightarrow P_\infty^{\square,1} \rightarrow P_\infty^{\square,2} \rightarrow 0$$

of finitely generated  $S_\infty^\square$ -modules which is a projective resolution of  $H^2(P_\infty^{\square,*})$ . Moreover, the cohomology of  $P_\infty^{\square,*}$  carries an action of  $R_\infty \hat{\otimes}_{\mathcal{O}} S_\infty^\square$ . We have therefore an isomorphism of  $R_\infty \otimes_{\mathcal{O}} \mathcal{O}/(\varpi^m)$ -modules:

$$\mathrm{Tor}_1^{S_\infty^\square}(H^2(P_\infty^{\square,*}), \mathcal{O}/(\varpi^m)) \simeq H^1(P_\infty^{\square,*} \otimes_{S_\infty^\square} \mathcal{O}/(\varpi^m)).$$

Observe that the action of  $R_\infty$  on  $H^2(P_\infty^{\square,*})$  factors through the quotient by  $\mathcal{I}R_\infty$  since the top-degree cohomology  $H^2(P_\infty^*)$  is constructed by patching the duals of a suitable system of  $H^0(D_{M_i}^* \otimes_{S_{M_i}} S_{N_i}/(\varpi^{N_i}))$  for various  $M_i \geq N_i \geq 1$  (cf. proof of [CG12<sup>+</sup>, Theorem 6.3]), and these  $H^0$  are all supported on the unramified locus by (5.14.1) and Proposition 5.5. So  $H^1(P_\infty^{\square,*} \otimes_{S_\infty^\square} \mathcal{O}/(\varpi^m)) \simeq H^1(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}_0}^\square$  is annihilated by  $\mathcal{I} \subset R_p$ . Similarly, computing  $\mathrm{Tor}_1^{S_\infty^\square}(H^2(P_\infty^{\square,*}), \mathcal{O})$  we see that  $H^1(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D))_{\mathfrak{m}_0}$  is annihilated by  $\mathcal{I}$ .

We conclude:

**Theorem 5.15.** *Assume that  $p > 3$ , that  $\bar{\rho}$  is Frobenius-distinguished at  $p$ , and that  $\bar{\rho}(G_F)$  contains  $\mathrm{SL}_2(\mathbb{F}_p)$ . Then the cohomology groups  $H^1(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}_0}$  and  $H^1(\mathrm{Sh}^{\mathrm{tor}}, \omega^1(-D))_{\mathfrak{m}_0}$  are supported on  $\mathrm{Spec}(R_p/\mathcal{I})$ , i.e., they give rise to Galois representations unramified at  $p$ .*

**5.16. A conjecture.** The conjecture below follows from the expected properties of modular Galois representations arising from forms of weight  $(1, \dots, 1)$ :

**Conjecture 5.17.** *Let  $F$  be a totally real number field of degree  $g$  over  $\mathbb{Q}$  and let  $p$  be an arbitrary prime number. Fix a prime  $\mathfrak{p}$  of  $F$  lying above  $p$ , and let  $\kappa_{\mathfrak{p}} = ((k_\tau)_{\tau \in \Sigma}, w)$  be a paritious weight such that  $k_\tau = 1$  for all  $\tau \in \Sigma_{\mathfrak{p}}$ . Let  $\mathbb{T}$  denote the image of the universal tame Hecke algebra acting on  $H^\bullet(\mathrm{Sh}^{\mathrm{tor}}, E/\mathcal{O} \otimes_{\mathcal{O}} \omega^{\kappa_{\mathfrak{p}}}(-D))$  and let  $\mathfrak{m}$  denote a non-Eisenstein maximal ideal of  $\mathbb{T}$ , with associated Galois representation  $\bar{\rho}$ . Then:*

- (1)  $\bar{\rho}$  is unramified at  $\mathfrak{p}$ ;
- (2) Let  $R_{\mathfrak{p}}$  denote the universal ring for framed  $\mathcal{O}$ -deformations of  $\bar{\rho}|_{G_{\mathfrak{p}}}$ , and let  $\mathcal{I}$  denote the proper ideal of  $R_{\mathfrak{p}}$  cutting out the locus of unramified lifts. Then there exists a positive integer  $n$  depending on  $g$  such that  $\mathcal{I}^n$  annihilates the  $R_{\mathfrak{p}}$ -module  $H^\bullet(\mathrm{Sh}^{\mathrm{tor}}, E/\mathcal{O} \otimes_{\mathcal{O}} \omega^{\kappa_{\mathfrak{p}}}(-D))_{\mathfrak{m}}$ .

Assuming the above conjecture and applying the arguments of the previous section one can prove that:

$$\mathrm{Tor}_i^{S_\infty^\square}(H^g(P_\infty^{\square,*}), \mathcal{O}/(\varpi^m)) \simeq H^i(P_\infty^{\square,*} \otimes_{S_\infty^\square} \mathcal{O}/(\varpi^m)) \simeq H^i(\mathrm{Sh}_m^{\mathrm{tor}}, \omega_m^1(-D))_{\mathfrak{m}}$$

for all  $i$  and all  $m$ . (Here the notation is as before). Using a suitable generalization of Proposition 5.5 to the case of non-Frobenius-distinguished representations, together with Grothendieck-Serre-Verdier duality, we see that the action of  $R_{\mathfrak{p}}$  on  $H^g(P_\infty^{\square,*})$  factors through  $\mathcal{I}$ . We then obtain:

**Theorem 5.18.** Fix a prime  $\mathfrak{p}$  of  $F$  above  $p > 3$  and a paritious weight  $\kappa_{\mathfrak{p}}$  with  $k_{\tau} = 1$  for all  $\tau \in \Sigma_{\mathfrak{p}}$ . Let  $\mathbb{T}$  denote the image of the universal tame Hecke algebra acting on  $H^{\bullet}(\mathrm{Sh}^{\mathrm{tor}}, E/\mathcal{O} \otimes_{\mathcal{O}} \omega^{\kappa_{\mathfrak{p}}}(-D))$  and let  $\mathfrak{m}$  denote a non-Eisenstein maximal ideal of  $\mathbb{T}$ . Assume the validity of Conjecture 5.17, and suppose Proposition 5.5 holds without the assumption of Frobenius distinguishness at  $\mathfrak{p}$ . Then  $H^{\bullet}(\mathrm{Sh}^{\mathrm{tor}}, E/\mathcal{O} \otimes_{\mathcal{O}} \omega^{\kappa_{\mathfrak{p}}}(-D))_{\mathfrak{m}}$  is supported on the unramified locus of  $\mathrm{Spec} R_{\mathfrak{p}}$ .

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MATTHEW EMERTON, DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF CHICAGO, 5734 S. UNIVERSITY AVENUE, CHICAGO, ILLINOIS 60637, USA

*E-mail address:* [emerton@math.uchicago.edu](mailto:emerton@math.uchicago.edu)

DAVIDE A. REDUZZI

*E-mail address:* [davide.reduzzi@yahoo.com](mailto:davide.reduzzi@yahoo.com)

LIANG XIAO, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, MATHEMATICAL SCIENCES BUILDING 210, UNIT 3009, STORRS, CONNECTICUT 06269, USA

*E-mail address:* [liang.xiao@uconn.edu](mailto:liang.xiao@uconn.edu)